Encryption III

601.442/642 Modern Cryptography

26th February 2026

Announcement

- Midterm grades released.

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- Homework 4 due **today**.

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- Homework 5 will be out today and due next Thursday (5th March).

Recap: History of Public-Key Encryption

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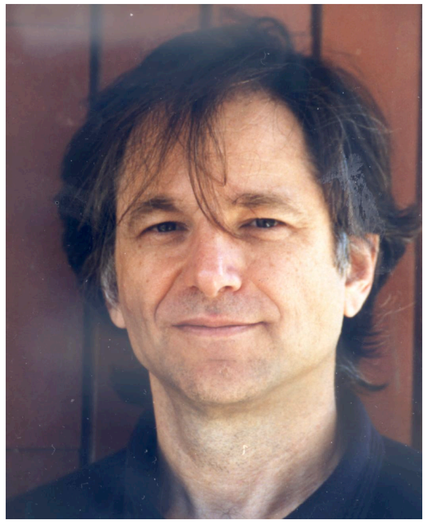

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- 1985: Taher ElGamal proposed a PKE based on Diffie-Hellman key exchange.
- 1997: GCHQ revealed that it had independently developed public-key cryptography between 1969-1974.
  - Public-key encryption, RSA and DH-like key exchange discovered by James Ellis, Clifford Cocks, and Malcolm Williamson.

Recap: ElGamal Encryption

ElGamal Encryption

- KeyGen(1^λ):
  - $(\mathbb{G}, g, p) \leftarrow \mathsf{GrpGen}(1^\lambda)$
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- Enc(pk, m):
  - $r \xleftarrow{\$} \{0,1\}^\lambda$
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Intuitively, security holds due to hardness of discrete log.
Formally, we require the stronger DDH assumption.

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Public-key encryption relies on computational problems that are **easy to compute** but **hard to reverse without** secret information.

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  - Multiplicative identity is 1.
- Computation over $\mathbb{Z}_N$ is **efficient**: Addition, subtraction, multiplication (and even exponentiation) takes $\mathsf{poly}(\log N) = \mathsf{poly}(\lambda)$ time.

Solving Equations over $\mathbb{Z}_N$

Goal: Understand hardness of computing cube-roots modulo N.

How hard is it to solve equations in $\mathbb{Z}_N$?

Solving Equations over $\mathbb{Z}_N$: Linear Equations

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- $\mathbb{Z}_N^\times = \{a \in \mathbb{Z}_N : \gcd(a, N) = 1\}$ forms a **group** under multiplication modulo N.
  - Identity is 1.
  - Each element has (multiplicative) inverse.

Solving Equations over $\mathbb{Z}_N$: Quadratic Equations

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- **Simplification:** Given a' solve for x such that

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Square Root modulo N

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  - Tonelli-Shanks algorithm can compute square-root modulo a prime N in $\mathsf{poly}(\log N)$ time.

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    - Up to 4 solutions modulo N.

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    - We can show that if an efficient algorithm can find square-root of a modulo N (even for just a uniformly random $a \in \mathbb{Z}_N$) then there exists an efficient algorithm for factoring N.
- When N is a product of distinct odd primes, computing square roots modulo N is equivalent to factoring N.

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But we no longer know if computing cube-roots is as hard as factoring.

That is, finding cube-roots might be easier.

RSA Assumption

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Let $\mathsf{GenRSA}(1^\lambda) \to (p, q)$ be a PPT algorithm that outputs two distinct λ-bit odd primes. Then for any non-uniform PPT adversary A

$$\Pr\left[A(N, e, x^e \bmod N) = x : \begin{array}{c}(p,q) \leftarrow \mathsf{GenRSA}(1^\lambda) \\ N = pq \\ e \xleftarrow{\$} \mathbb{Z}^\times_{\varphi(N)} \\ x \xleftarrow{\$} \mathbb{Z}^\times_N\end{array}\right] \leq \mathsf{negl}(\lambda)\,.$$

An efficient adversary cannot compute the e-th root modulo N.

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Next week's HW: converting textbook RSA to CPA-secure scheme.

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- Best known heuristic algorithm $2^{o\left(\sqrt[3]{\lambda \log^2 \lambda}\right)}$.
- Parameters for RSA
  - N is at least 2048 bits.
  - Recommended to use 3072-bit N.