Foundations

601.442/642 Modern Cryptography

3rd March 2026

The Picture So Far

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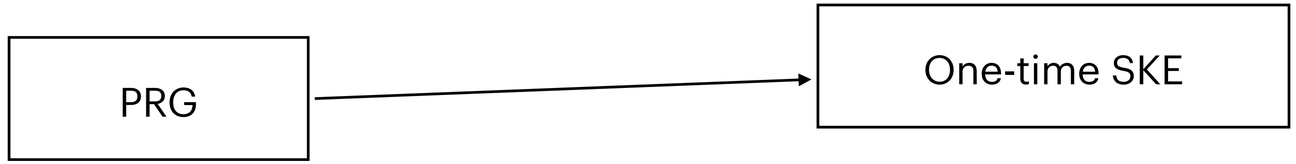

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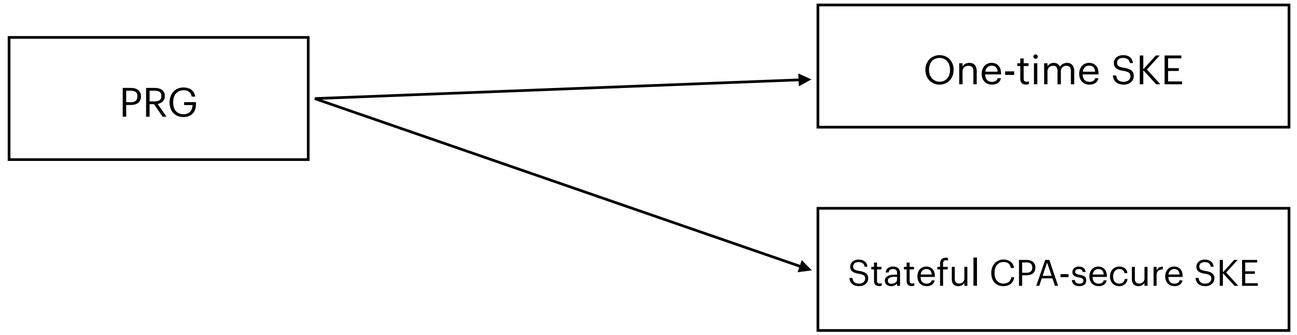

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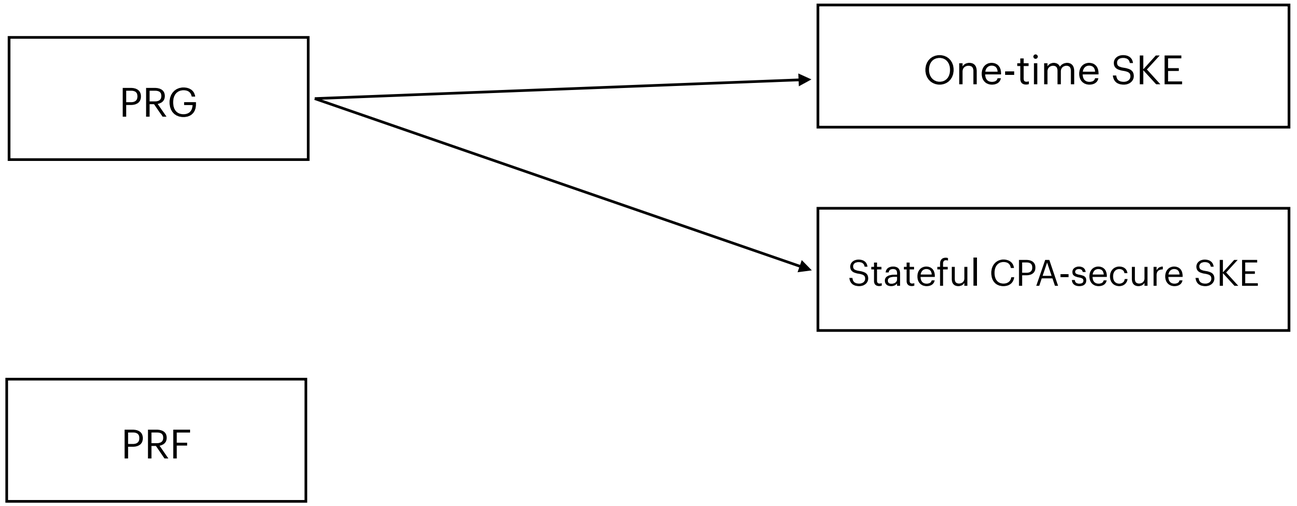

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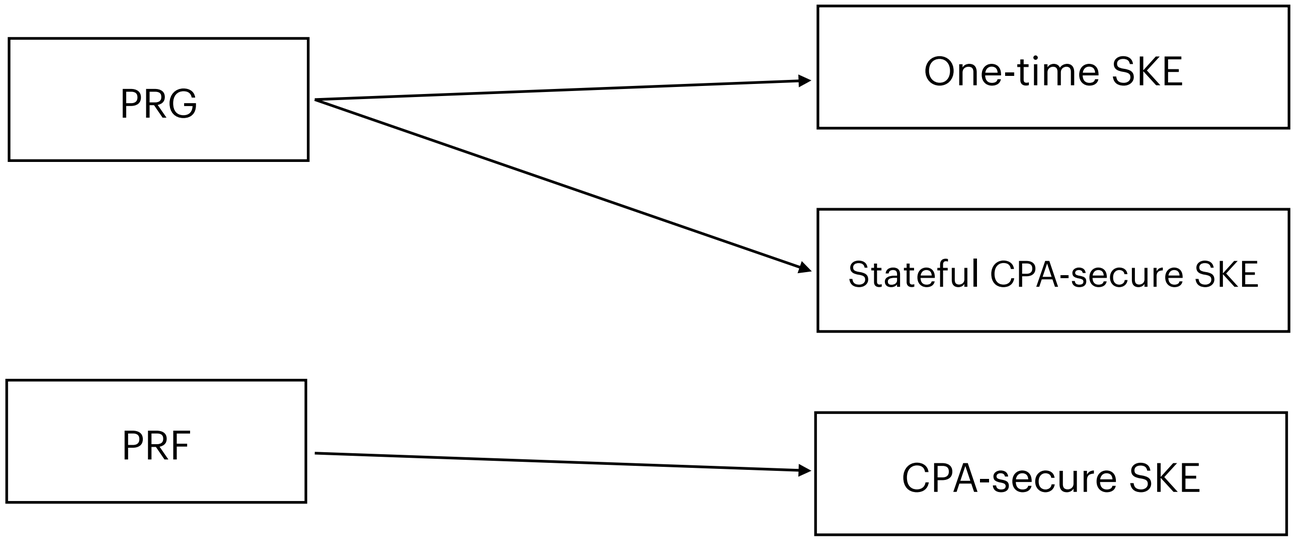

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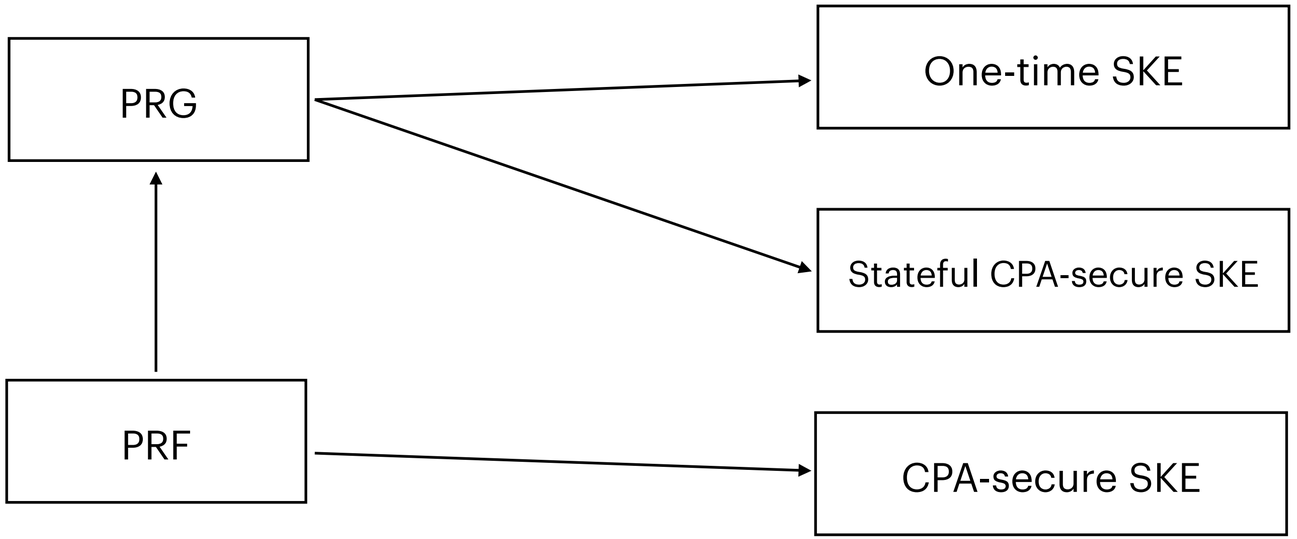

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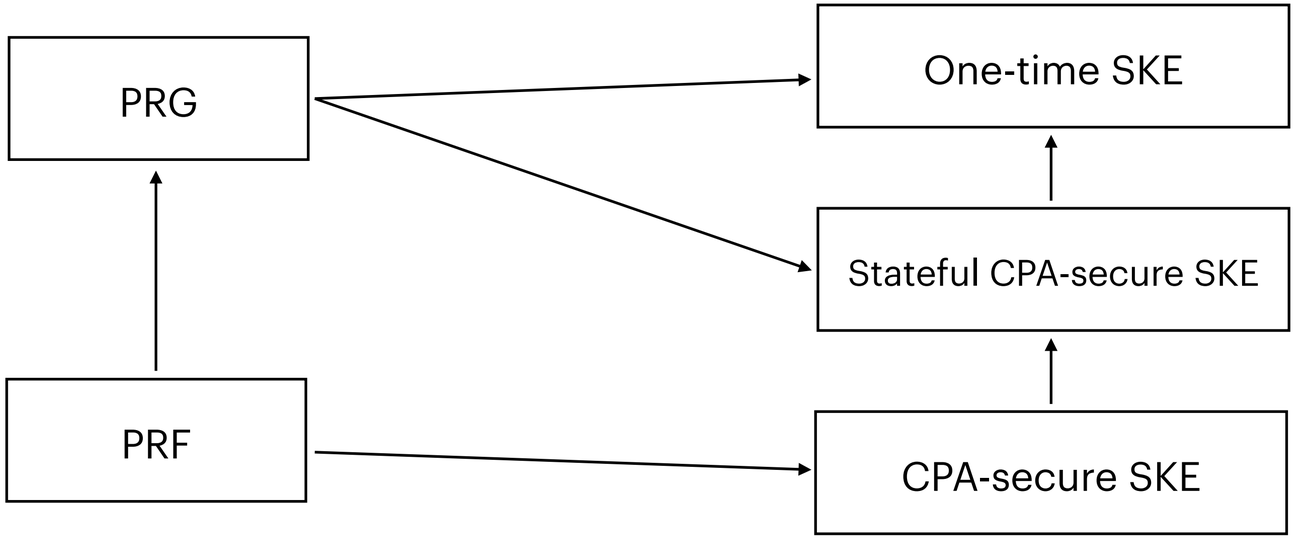

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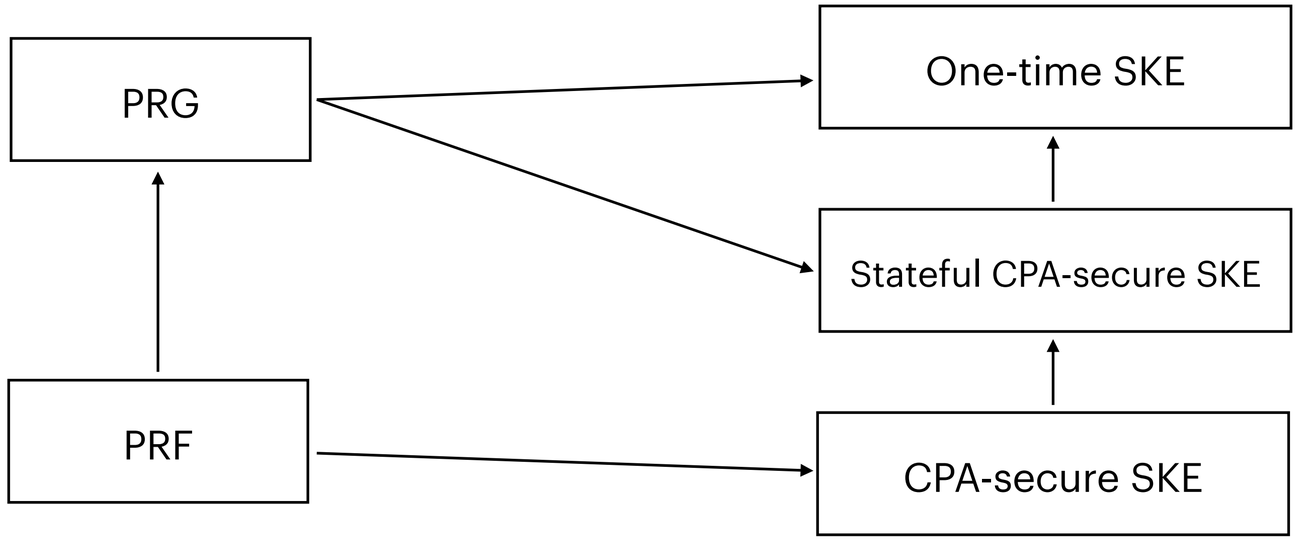
Why do we believe PRGs and PRFs exist?
How to construct them?



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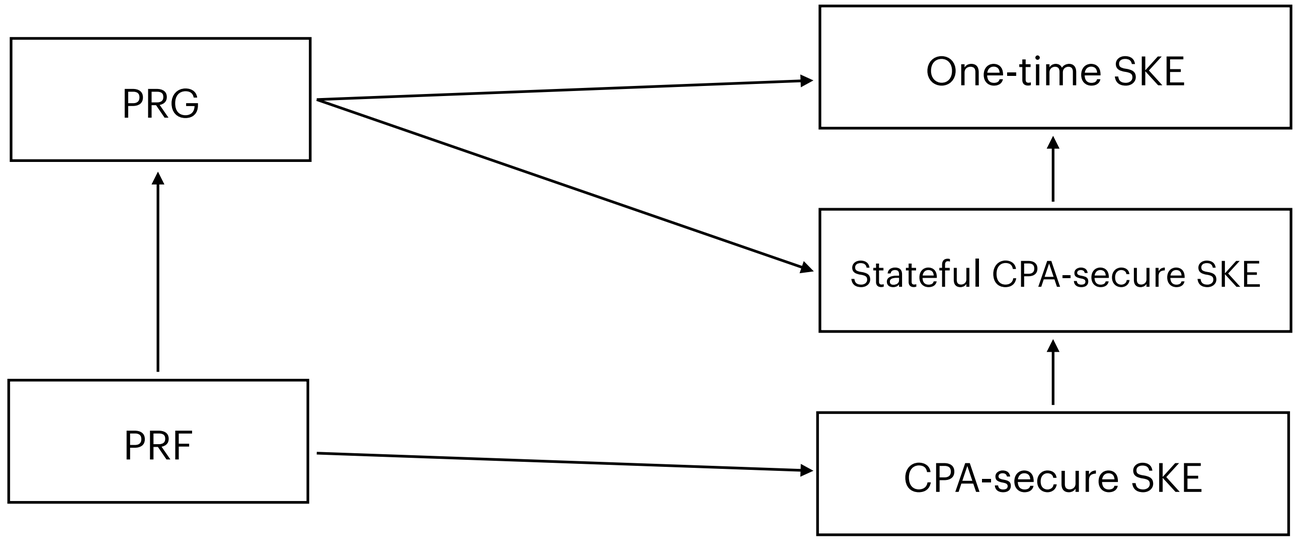

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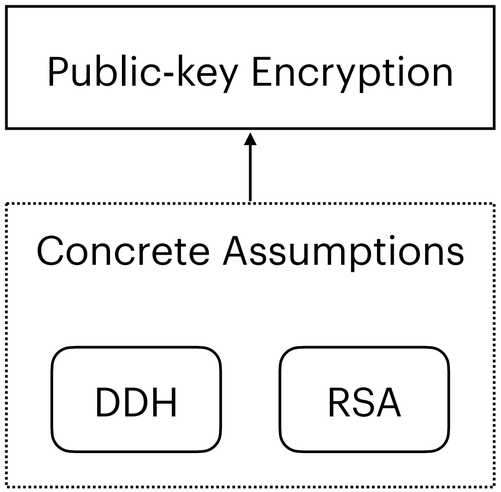

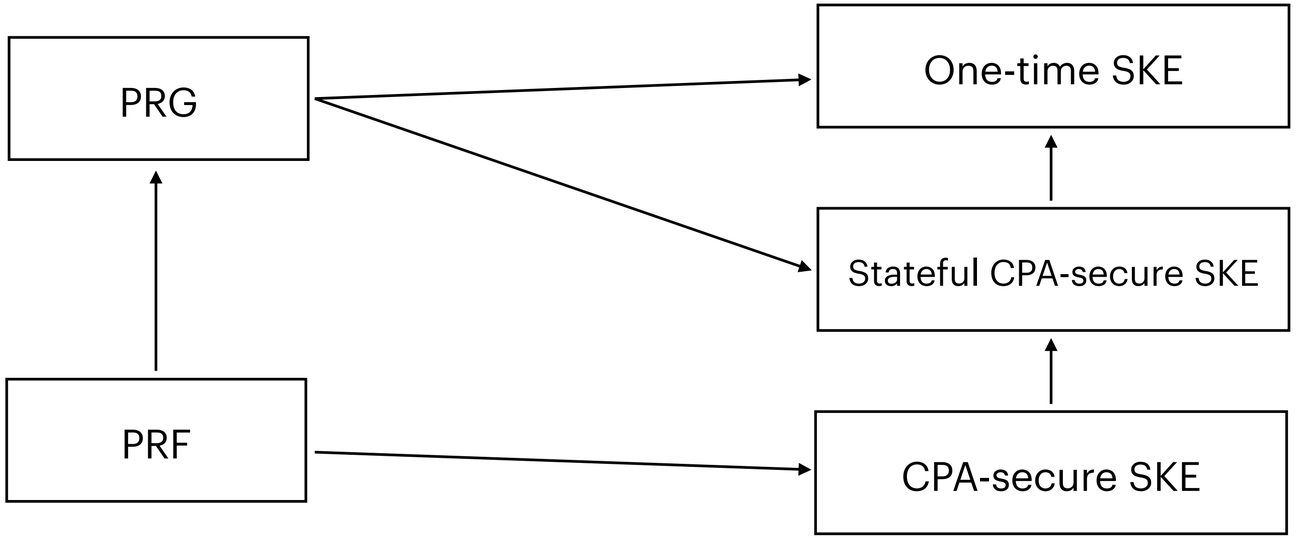

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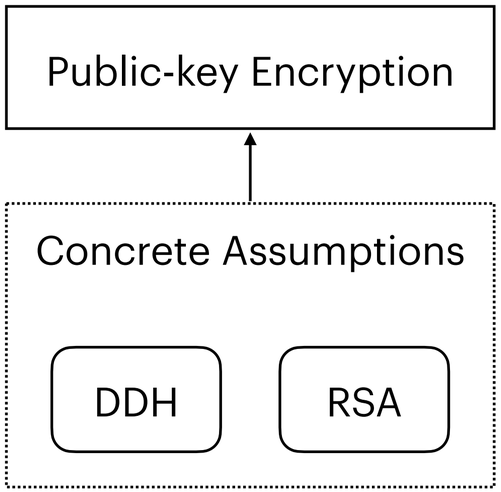

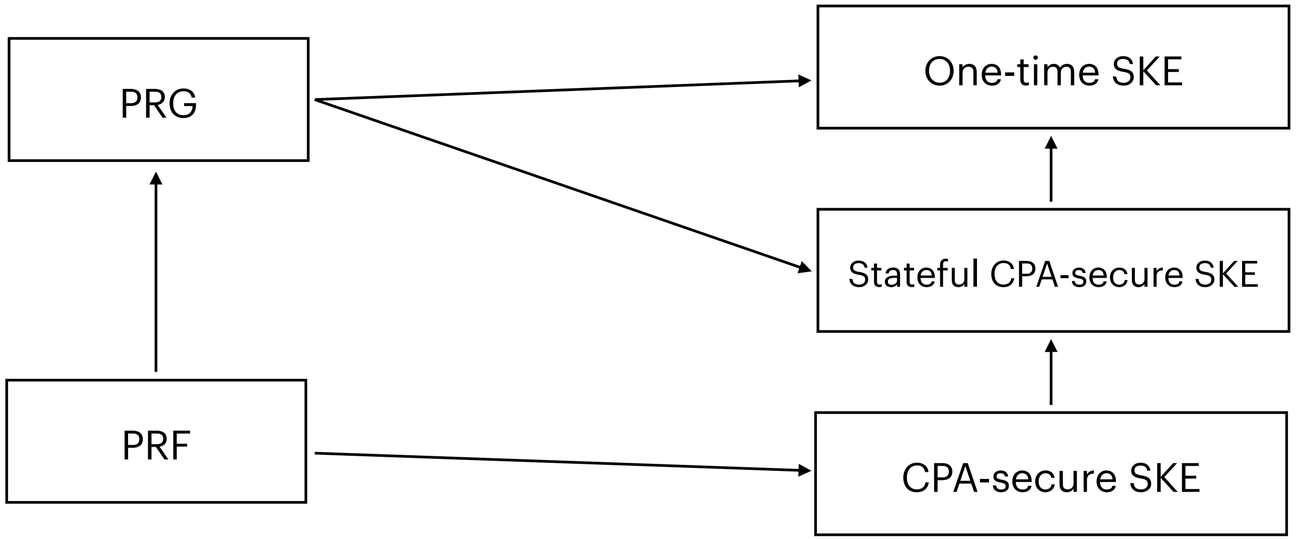

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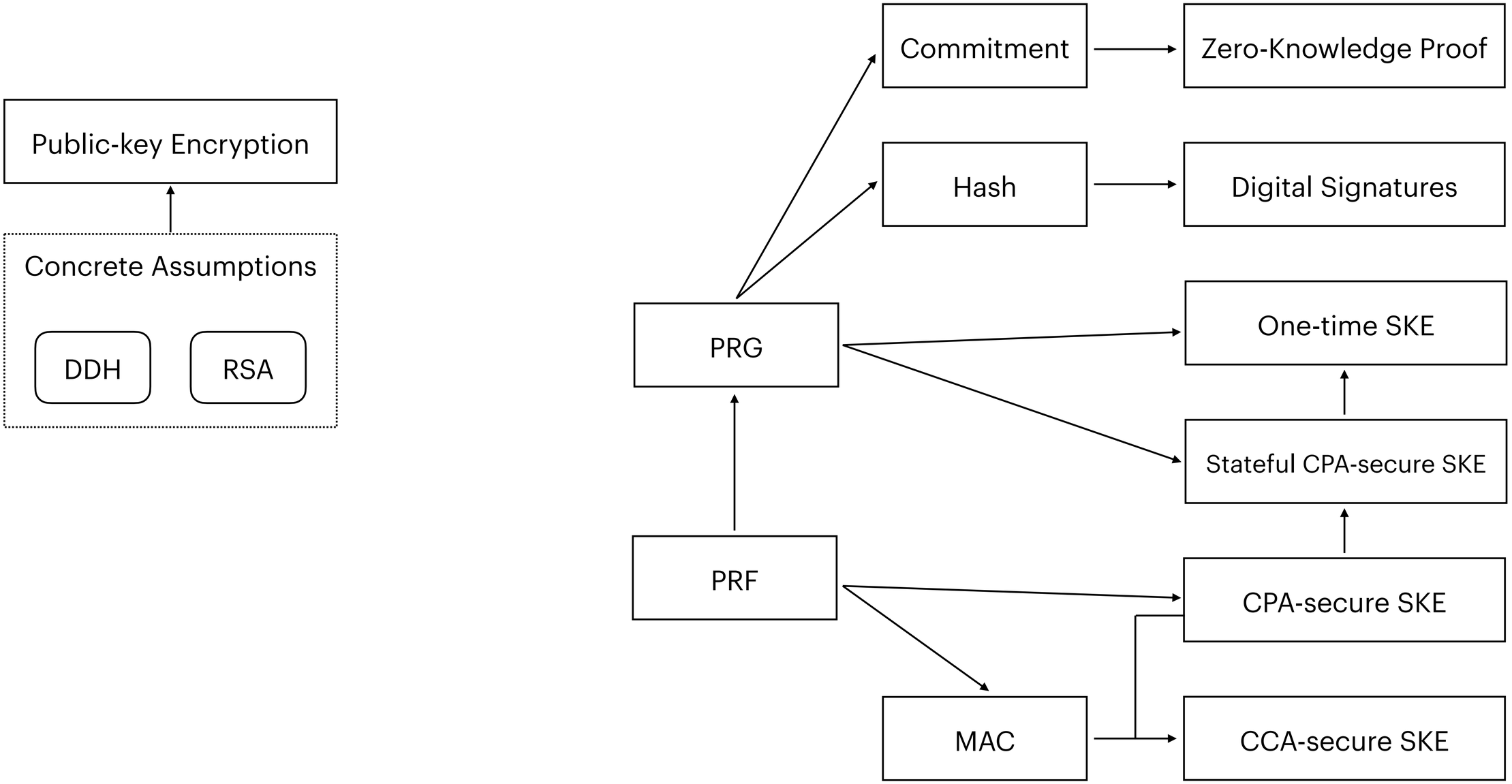
Can we construct PRFs from PRGs?

Can we construct PKE from PRFs?

The Cryptography Landscape



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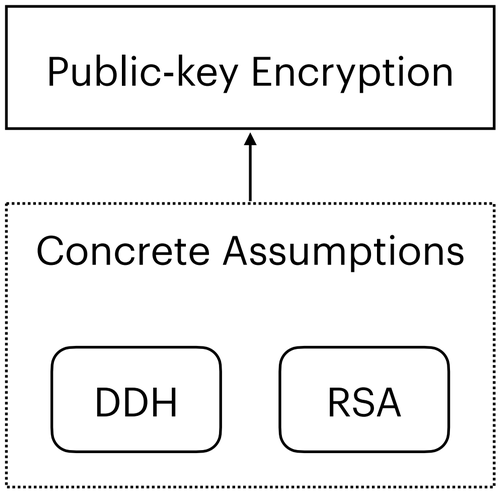
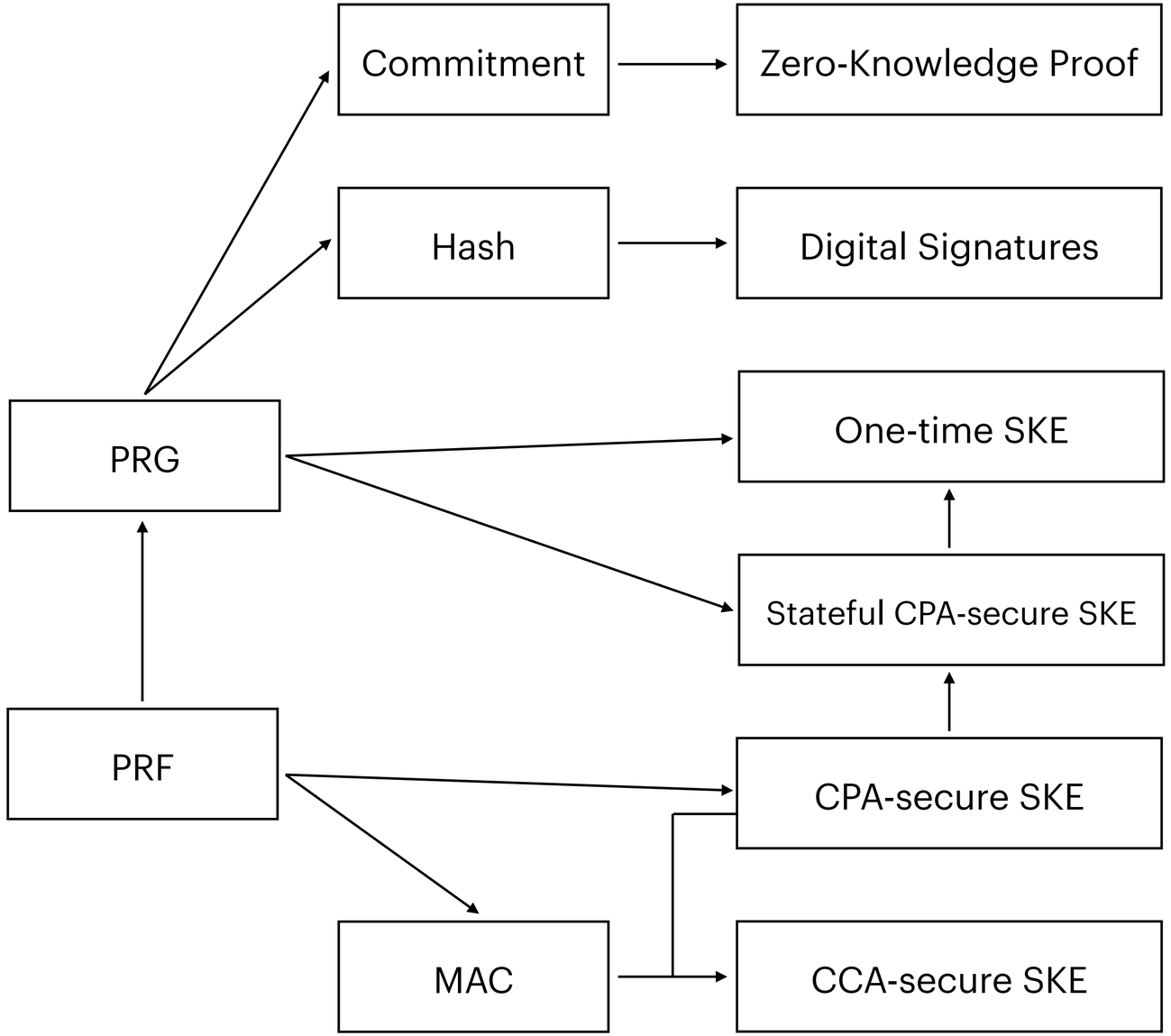


All results are of the form: $X \implies Y$.

What is the minimum assumption X we need to make?



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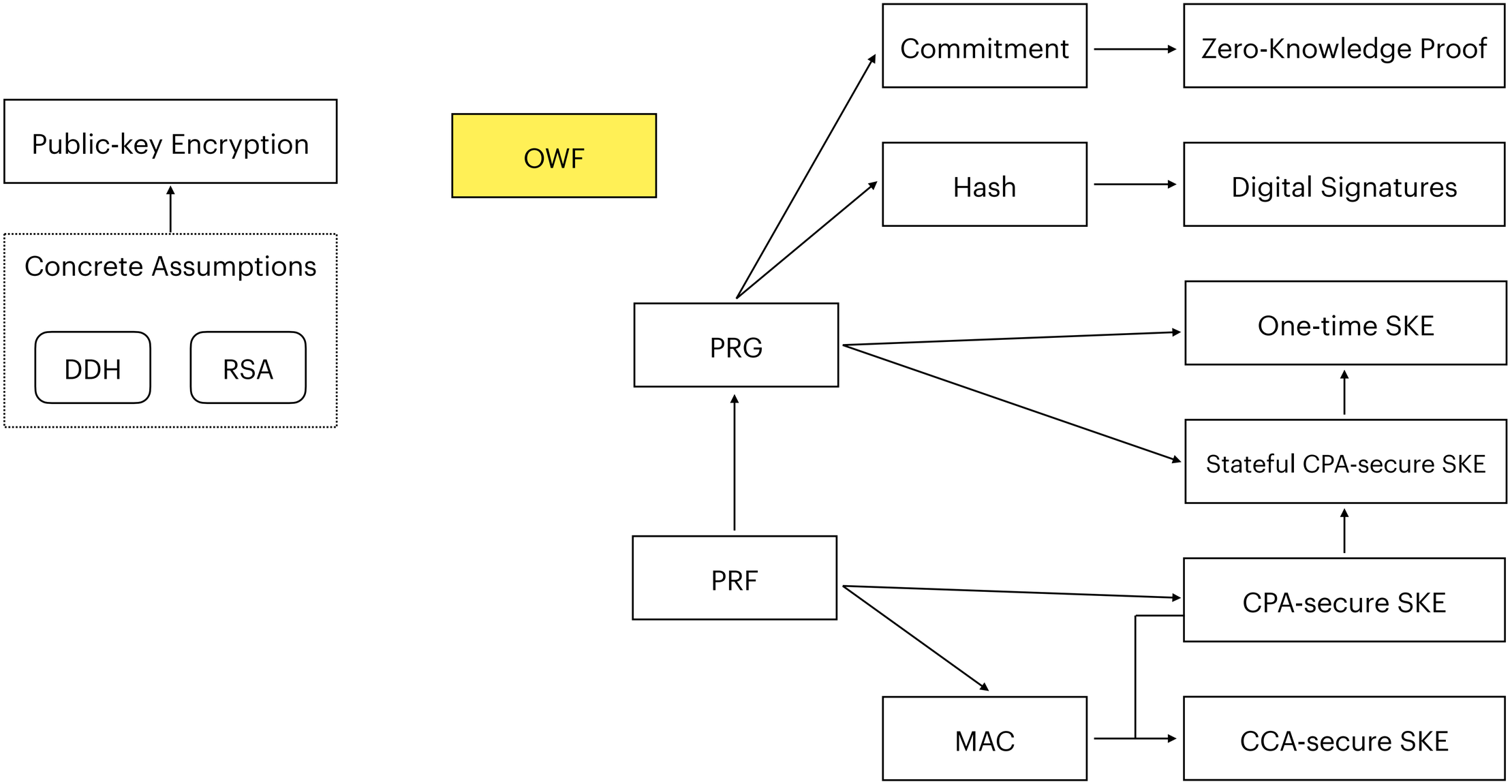

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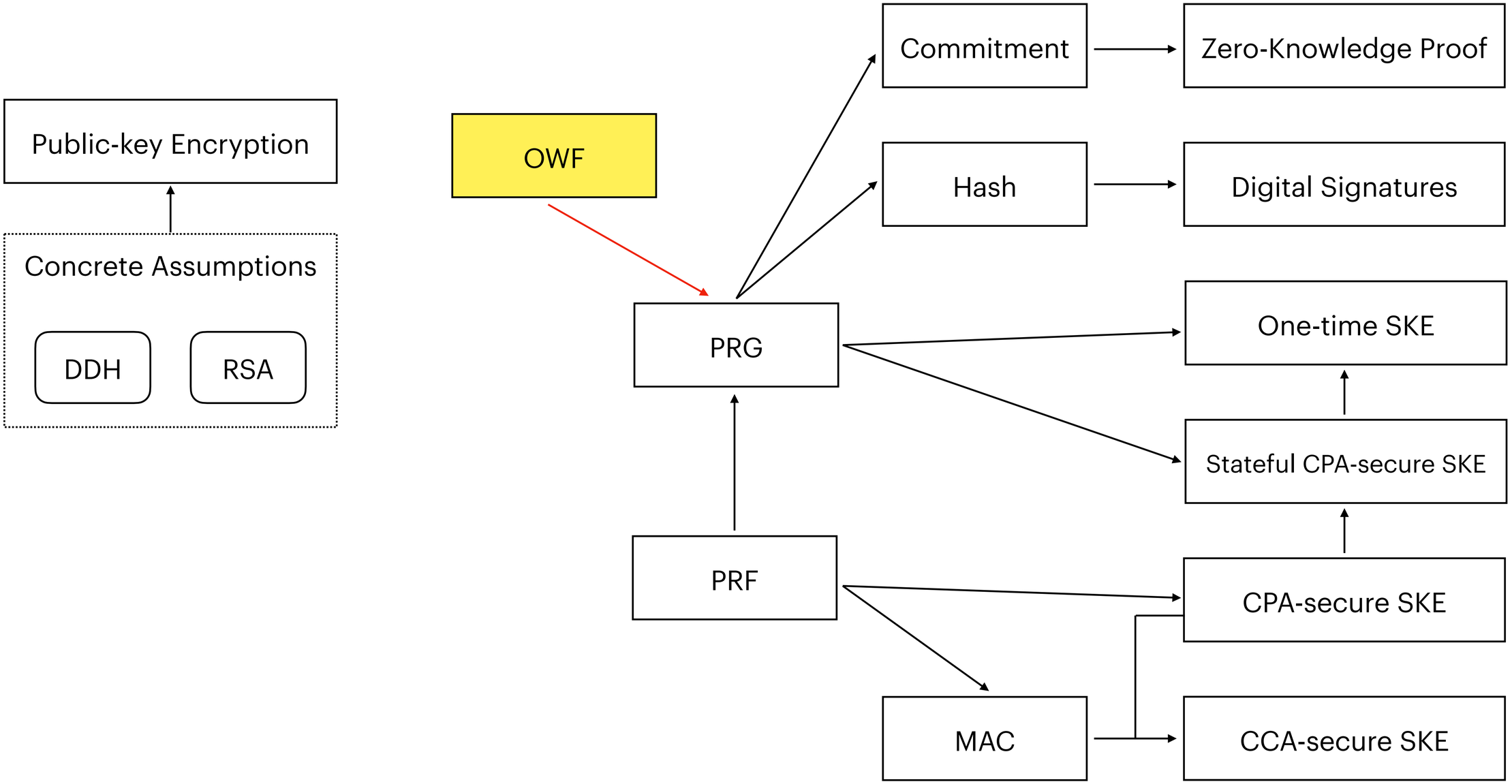

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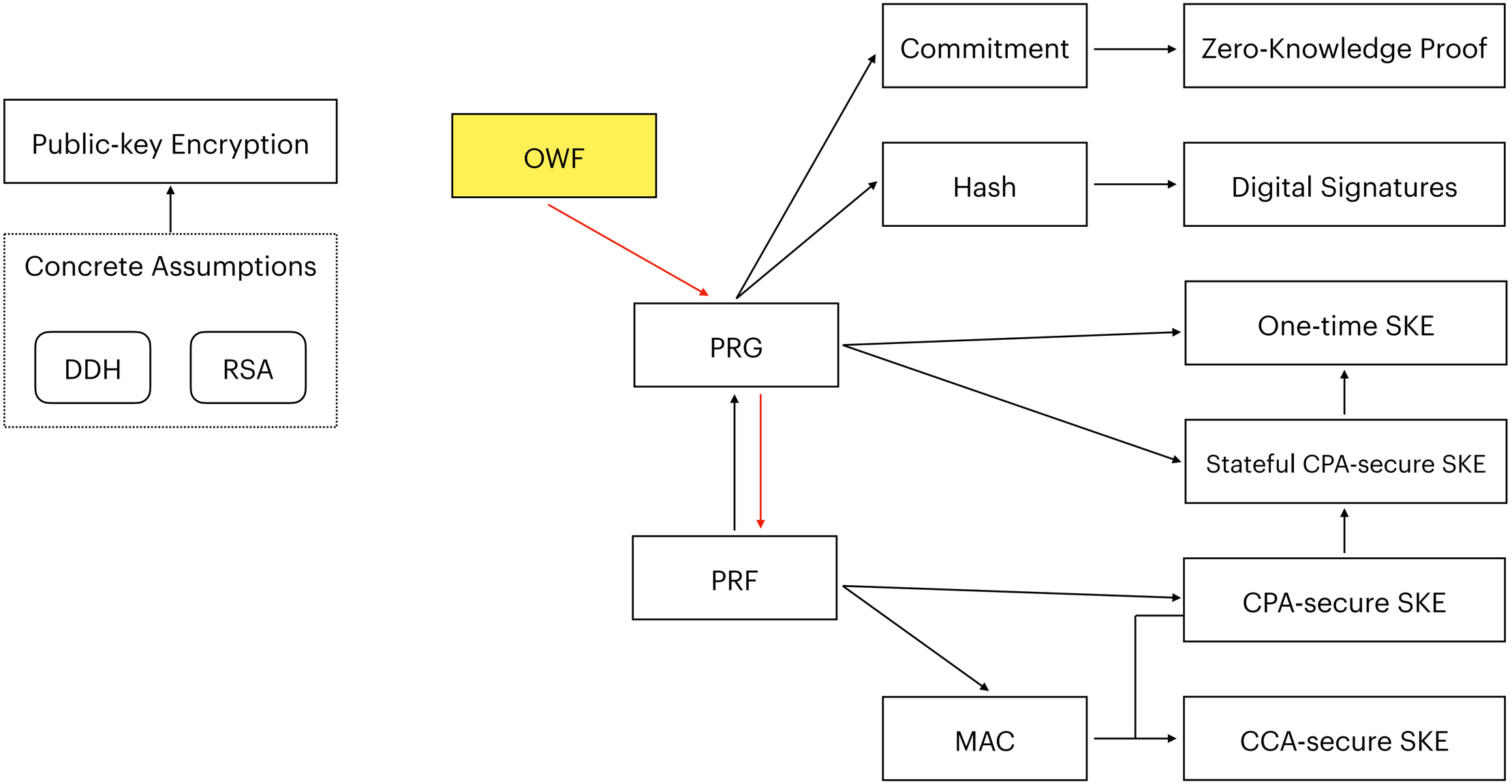

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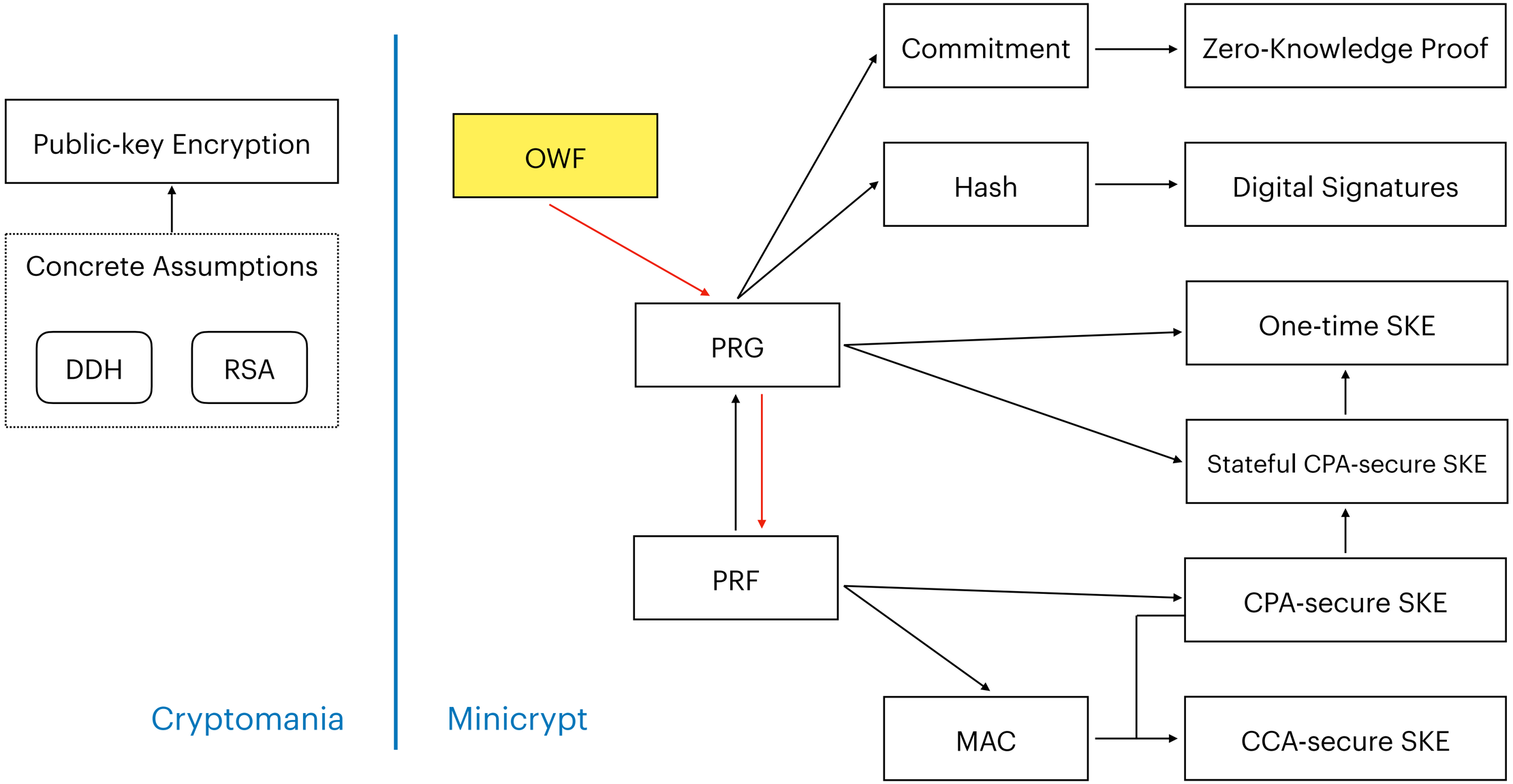

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Security stems from searching for a solution being hard.

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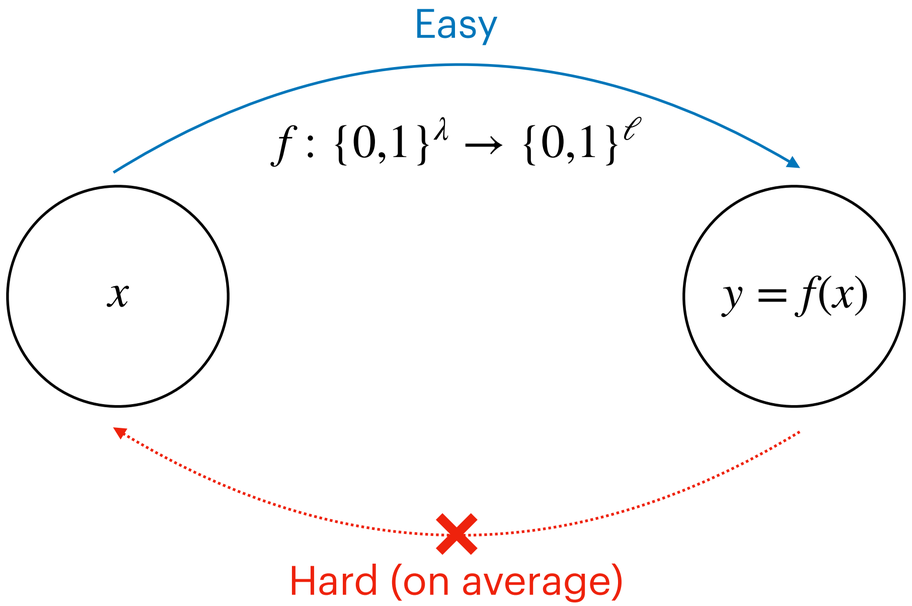
Security stems from searching for a solution being hard on average.

One-Way Functions



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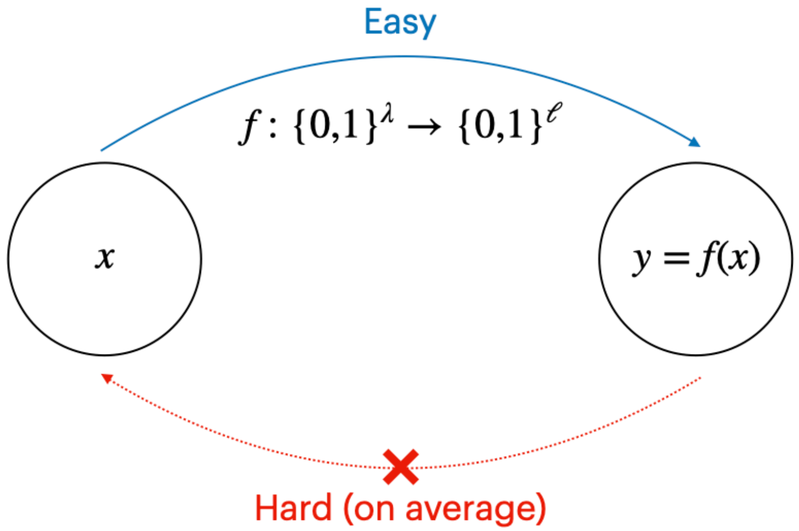

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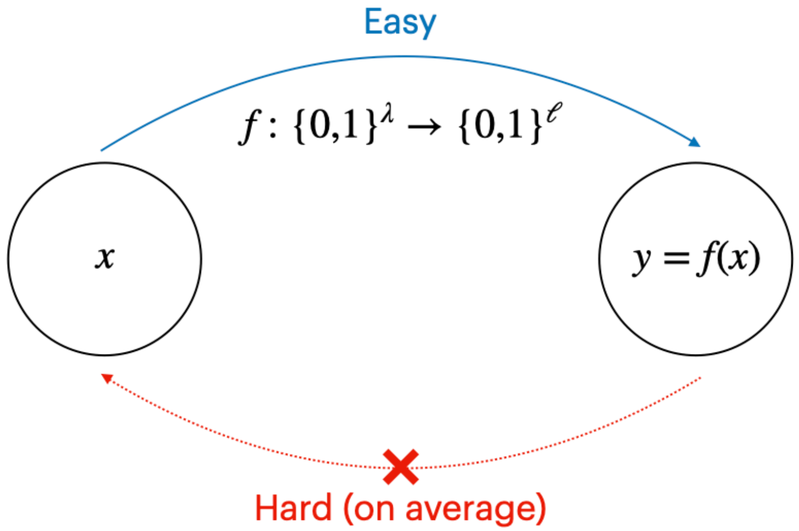

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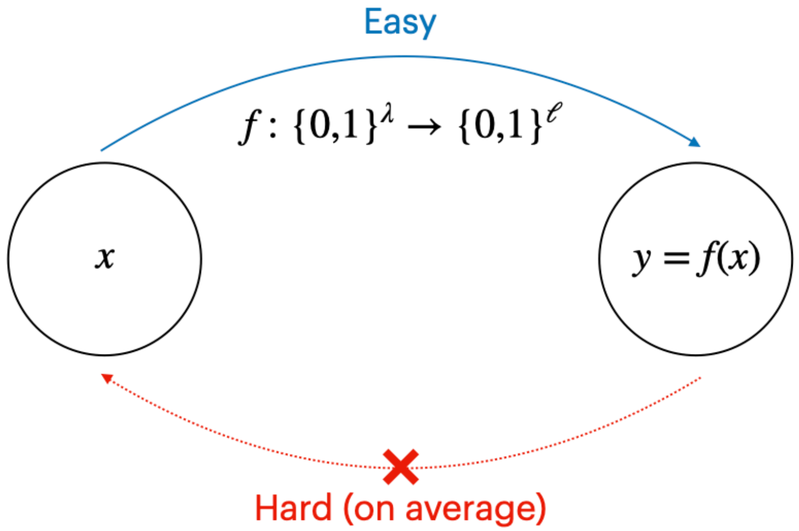

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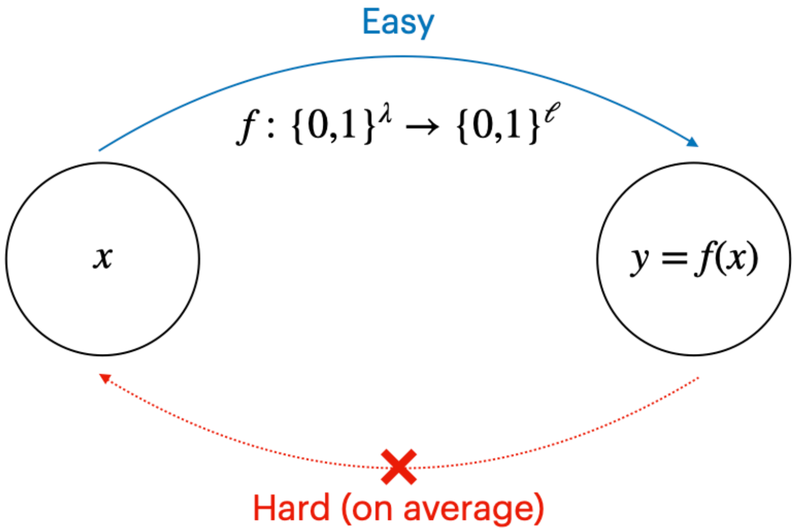

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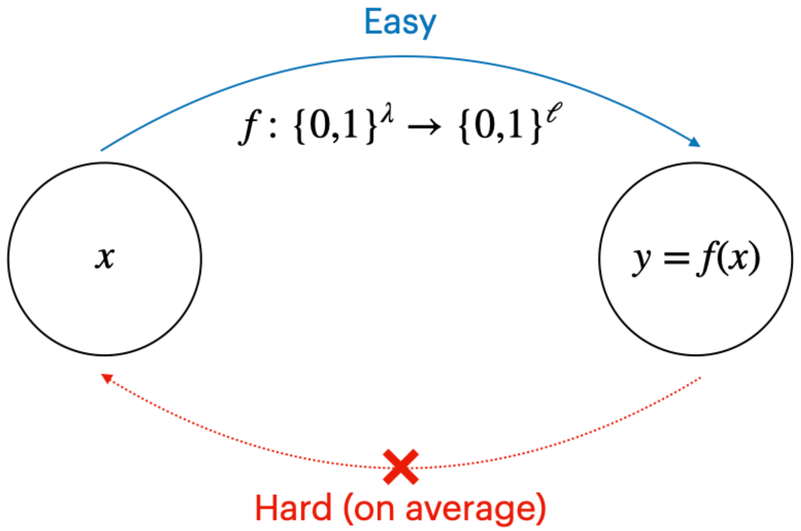

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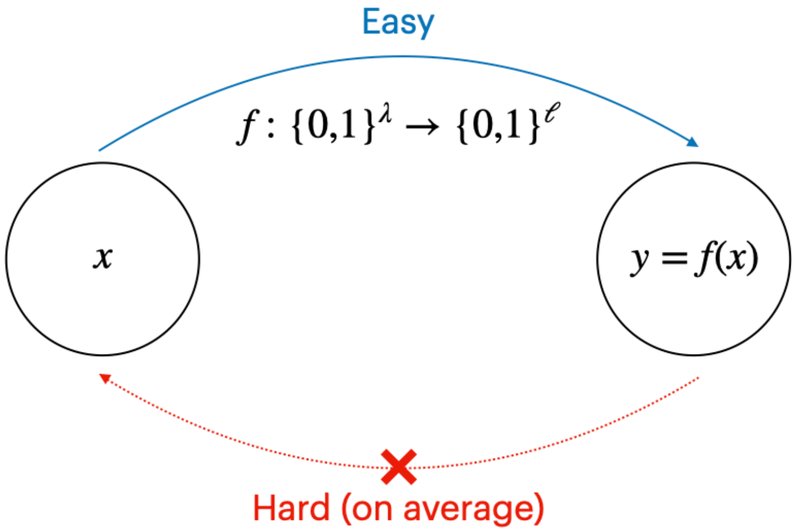

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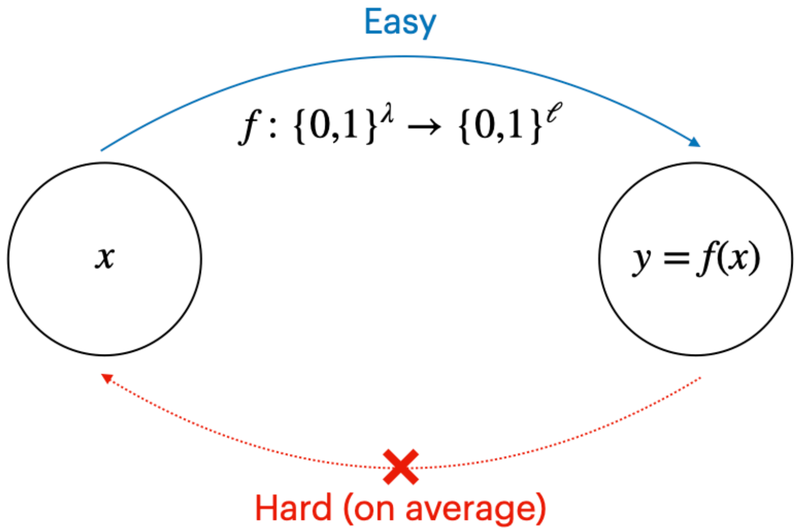

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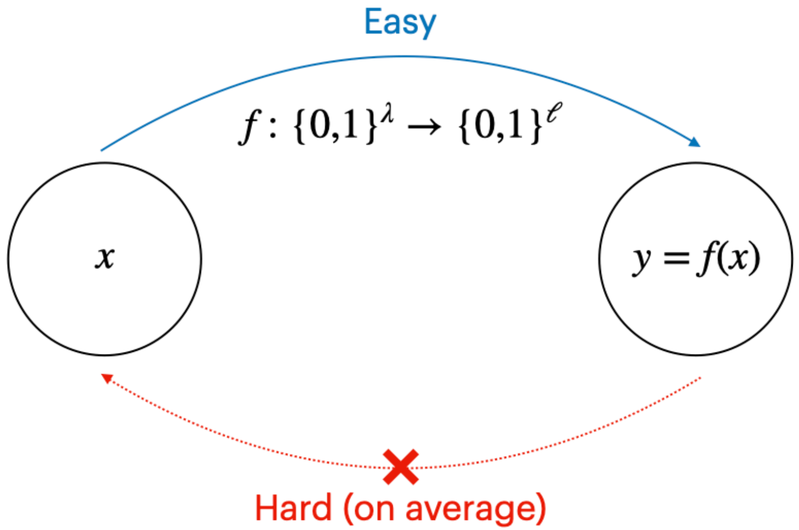

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We want to capture that it is hard to find **an** inverse of y.

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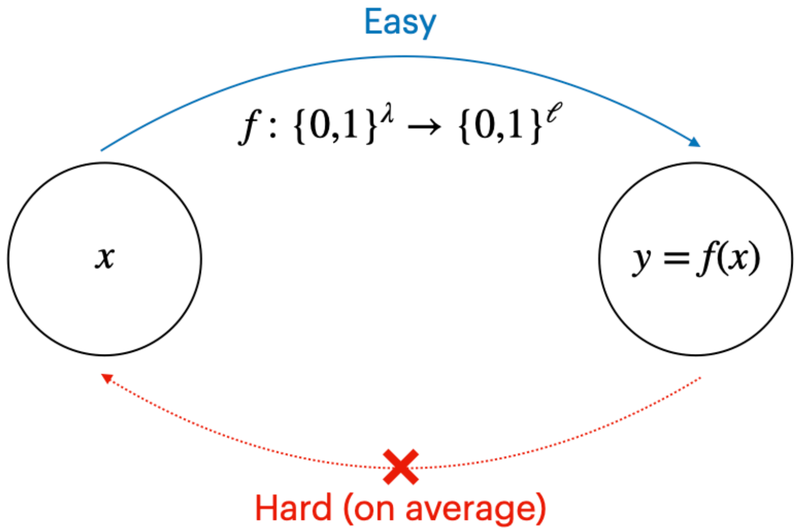

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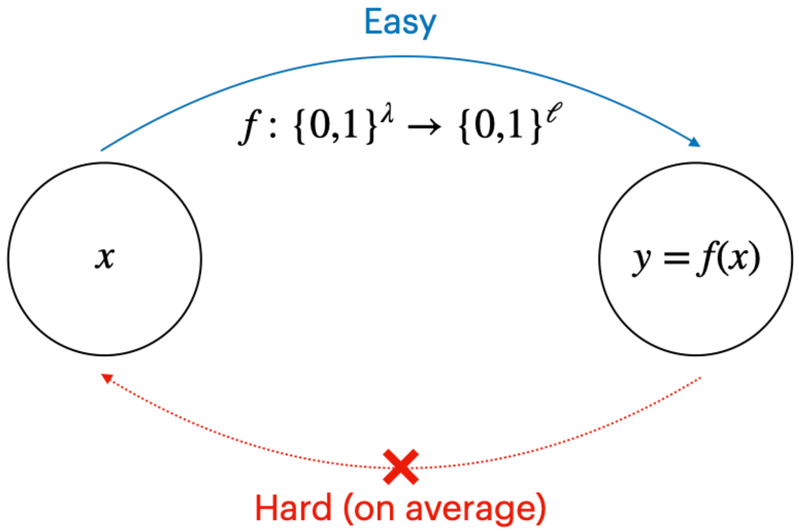

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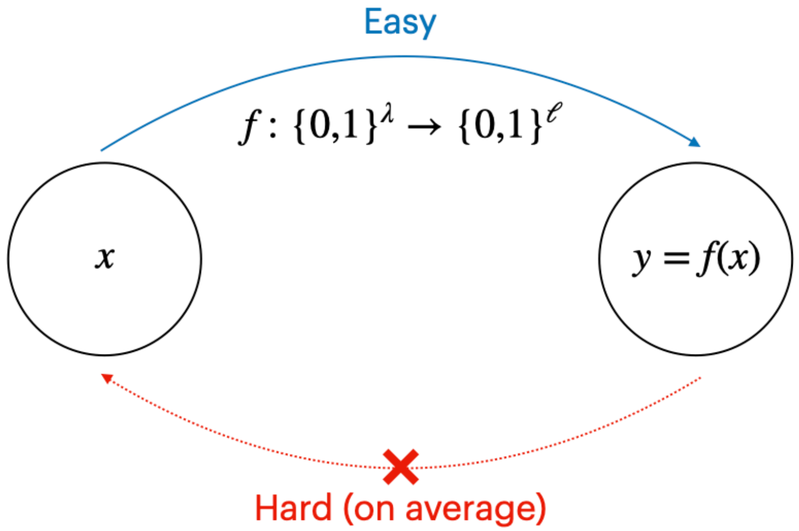

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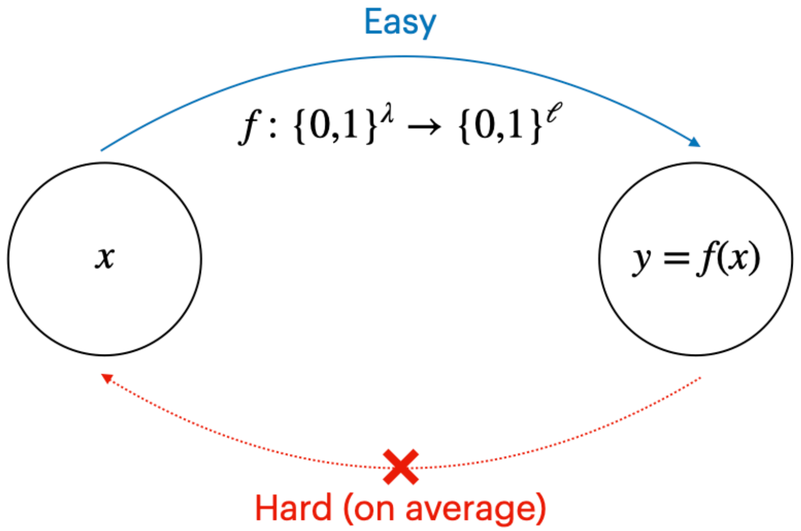

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Each image y has a unique pre-image x

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A family of functions $\{f_\lambda : \{0,1\}^\lambda \to \{0,1\}^{\ell}\}_\lambda$ is a **one-way function** if it satisfies the following properties.

- **Easy to compute:** For all $\lambda \in \mathbb{N}$, f_λ can be computed in polynomial time.
- **Hard to invert:** For all NUPPT adversaries A, there exists a negligible function **negl**, such that for all $\lambda \in \mathbb{N}$

$$\Pr\left[f(x') = y : \begin{array}{c} x \xleftarrow{\$} \{0,1\}^\lambda \\ y := f(x) \\ x' \leftarrow A(1^\lambda, y) \end{array}\right] \leq \mathsf{negl}(\lambda).$$

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- **Efficient Verification:** Given x' it is efficient to verify if $f(x') = y$.

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Discrete Logarithm Assumption

Let $(G, \cdot)$ be a cyclic group of order p with generator g, then for every NUPPT adversary A, there exists a negligible function negl such that

$$\Pr\left[a = a' : \begin{array}{c} a \xleftarrow{\$} \mathbb{Z}_p \\ a' \leftarrow A(G, p, g, g^a) \end{array}\right] \leq \text{negl}(\lambda)\,.$$

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RSA Assumption

Let $\mathsf{GenRSA}(1^\lambda) \to (p, q)$ be a PPT algorithm that outputs two distinct λ-bit odd primes. Then for any non-uniform PPT adversary A

$$\Pr\left[A(N, e, x^e \bmod N) = x : \begin{array}{c}(p,q) \leftarrow \mathsf{GenRSA}(1^\lambda) \\ N = pq \\ e \xleftarrow{\$} \mathbb{Z}^\times_{\varphi(N)} \\ x \xleftarrow{\$} \mathbb{Z}^\times_N\end{array}\right] \leq \mathsf{negl}(\lambda)\,.$$

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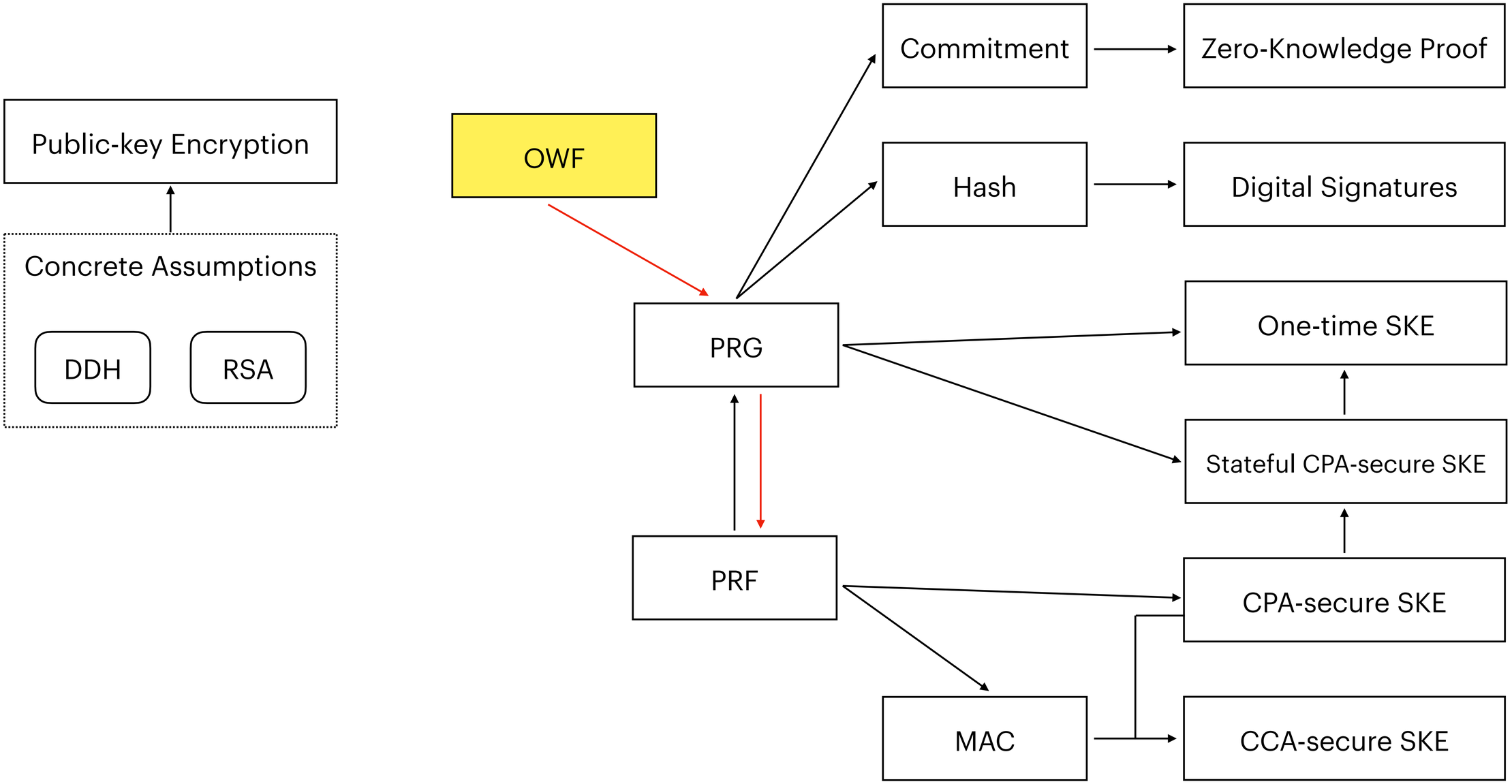
    - But $P \neq NP$ is not sufficient to argue existence of OWF ($P \neq NP$ captures worst-case hardness, OWF requires average-case hardness).
- In general, most cryptographic primitives imply OWF.

The Cryptography Landscape



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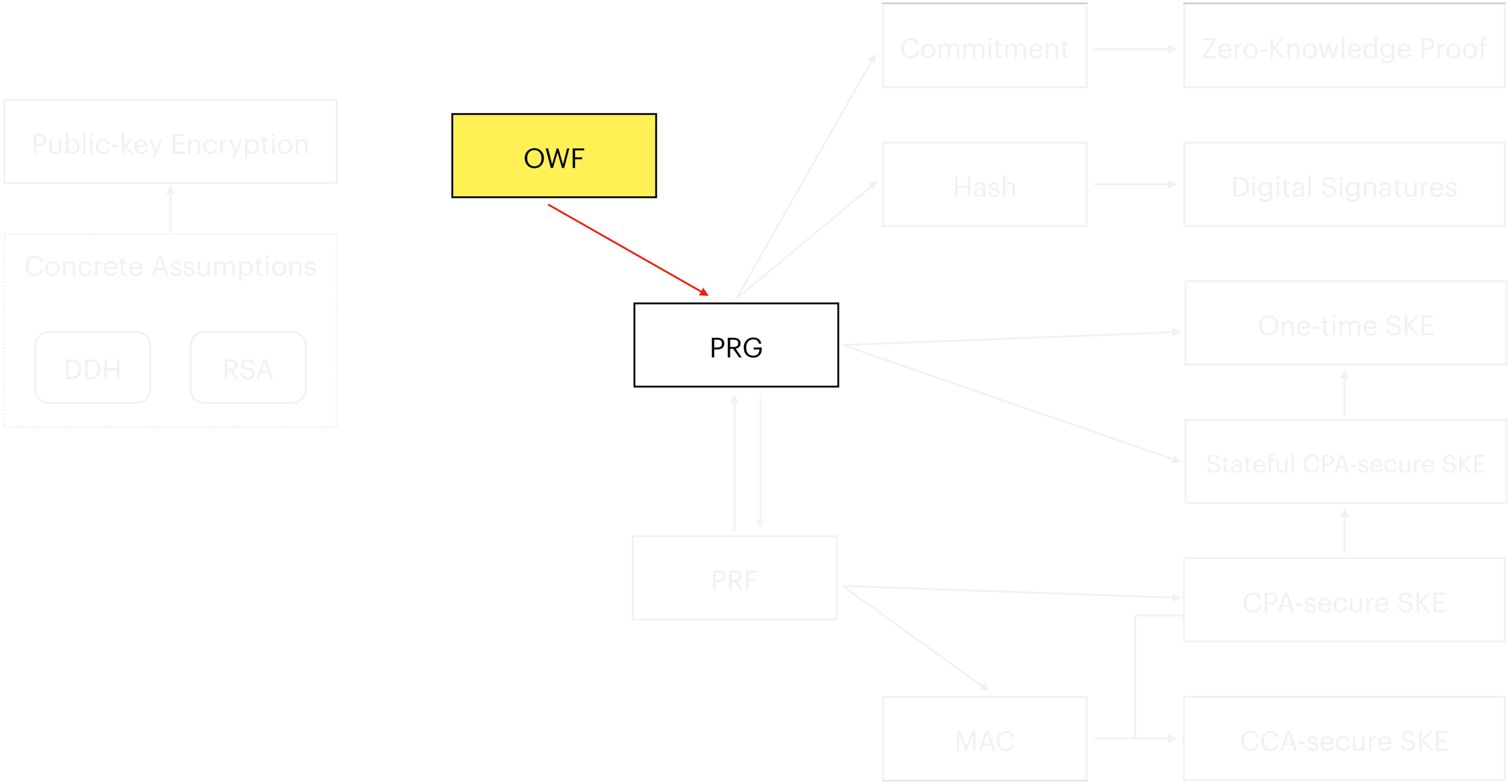

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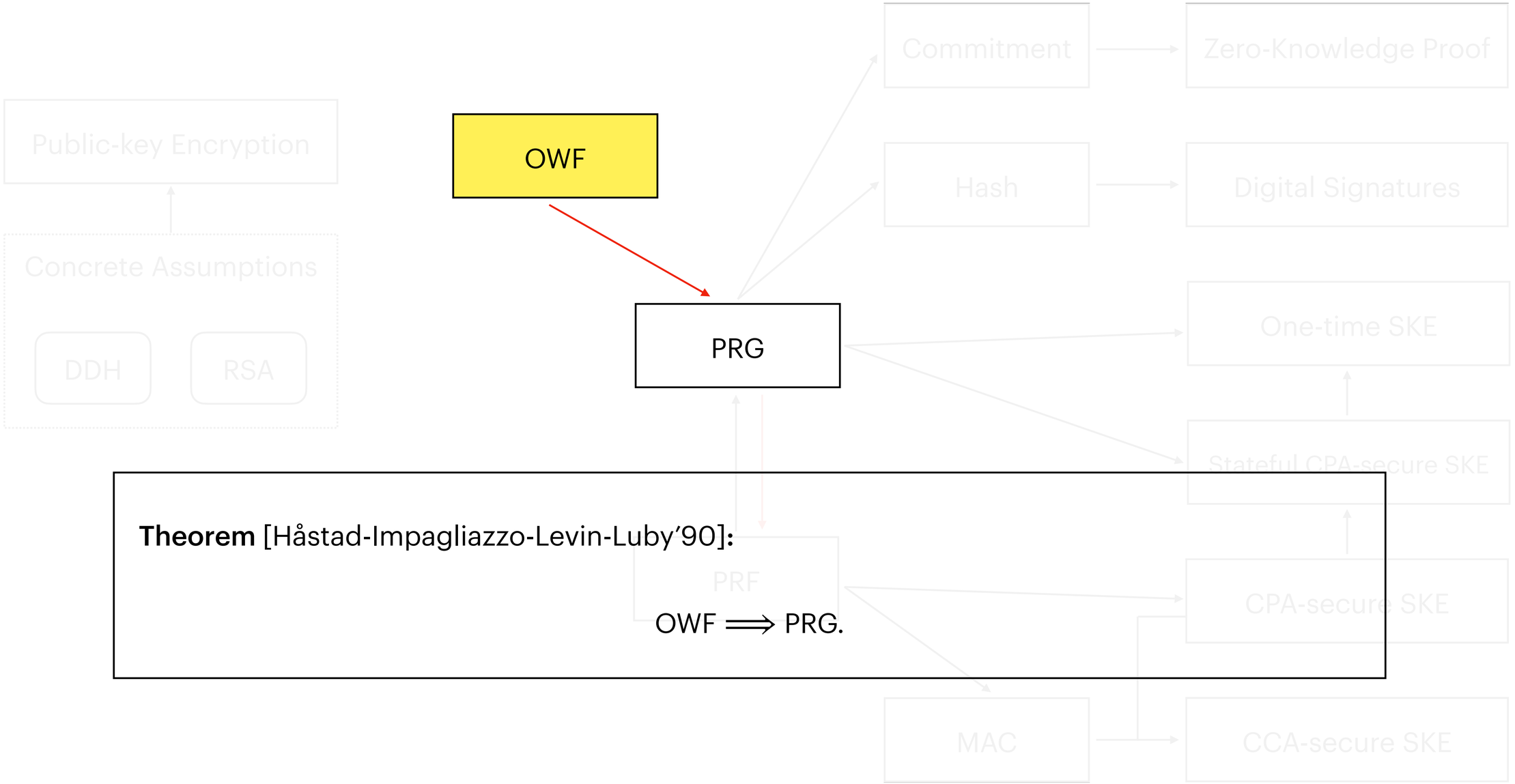

Theorem [Håstad-Impagliazzo-Levin-Luby'90]**:**

OWF $\Longrightarrow$ PRG.

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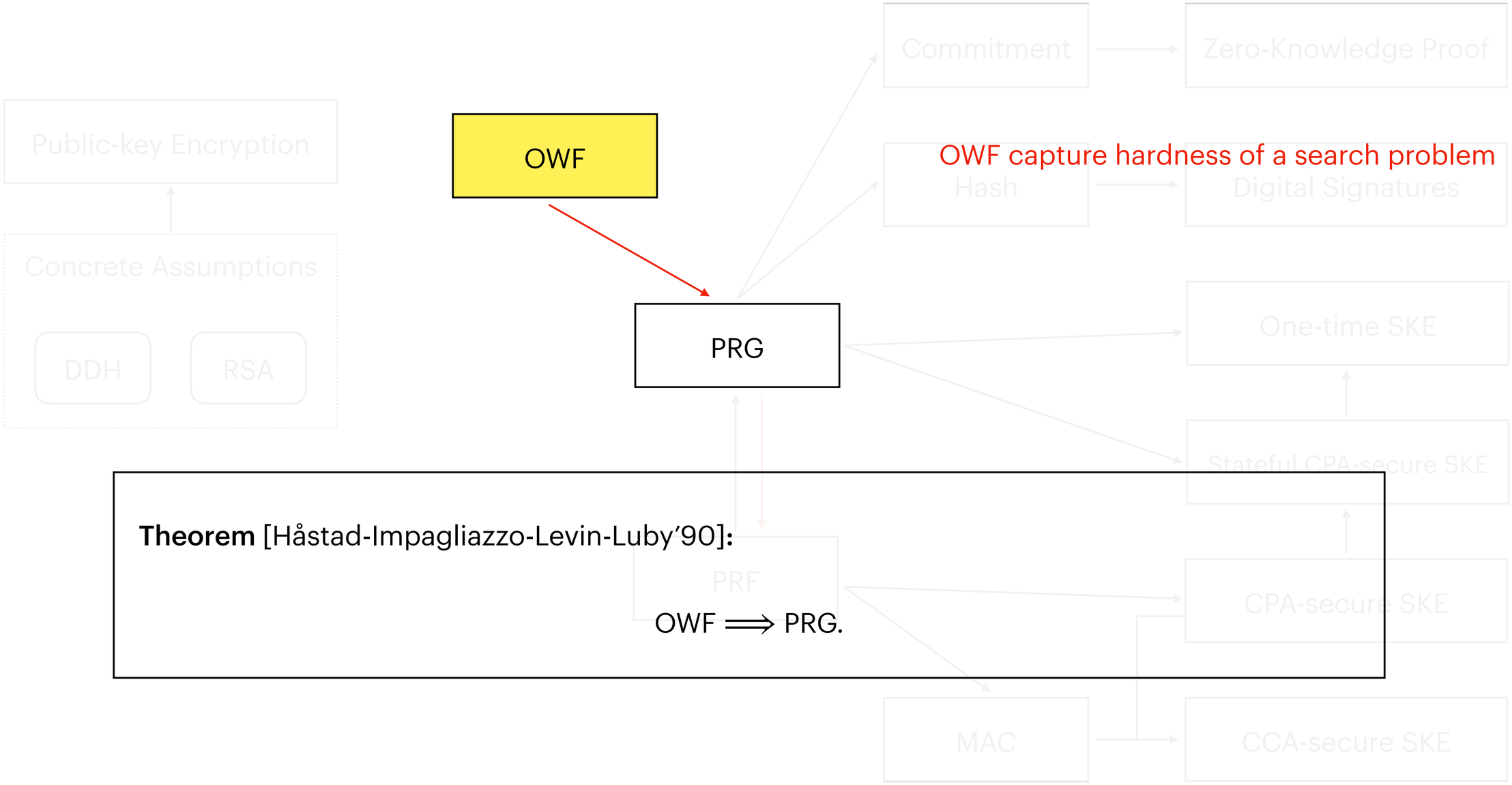

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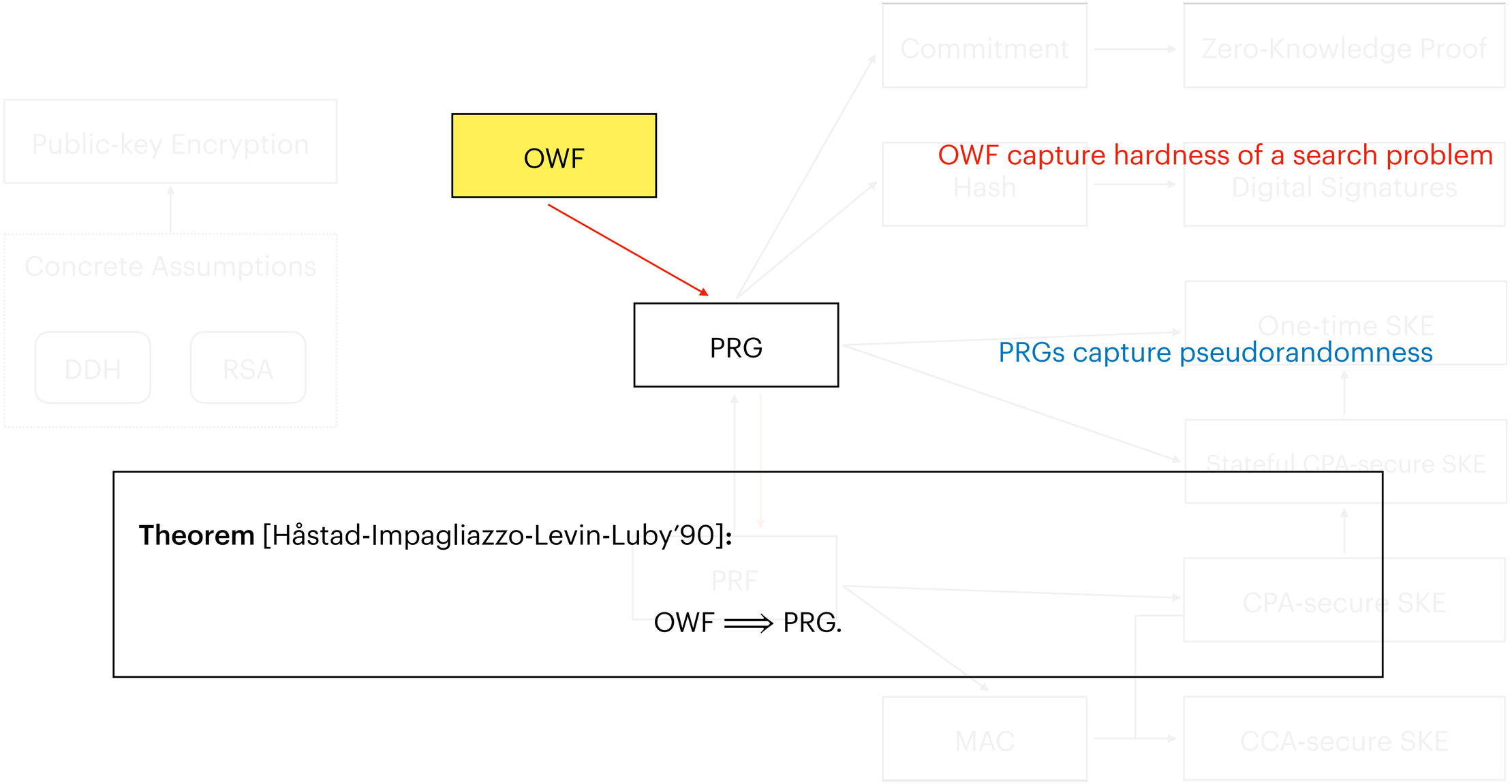

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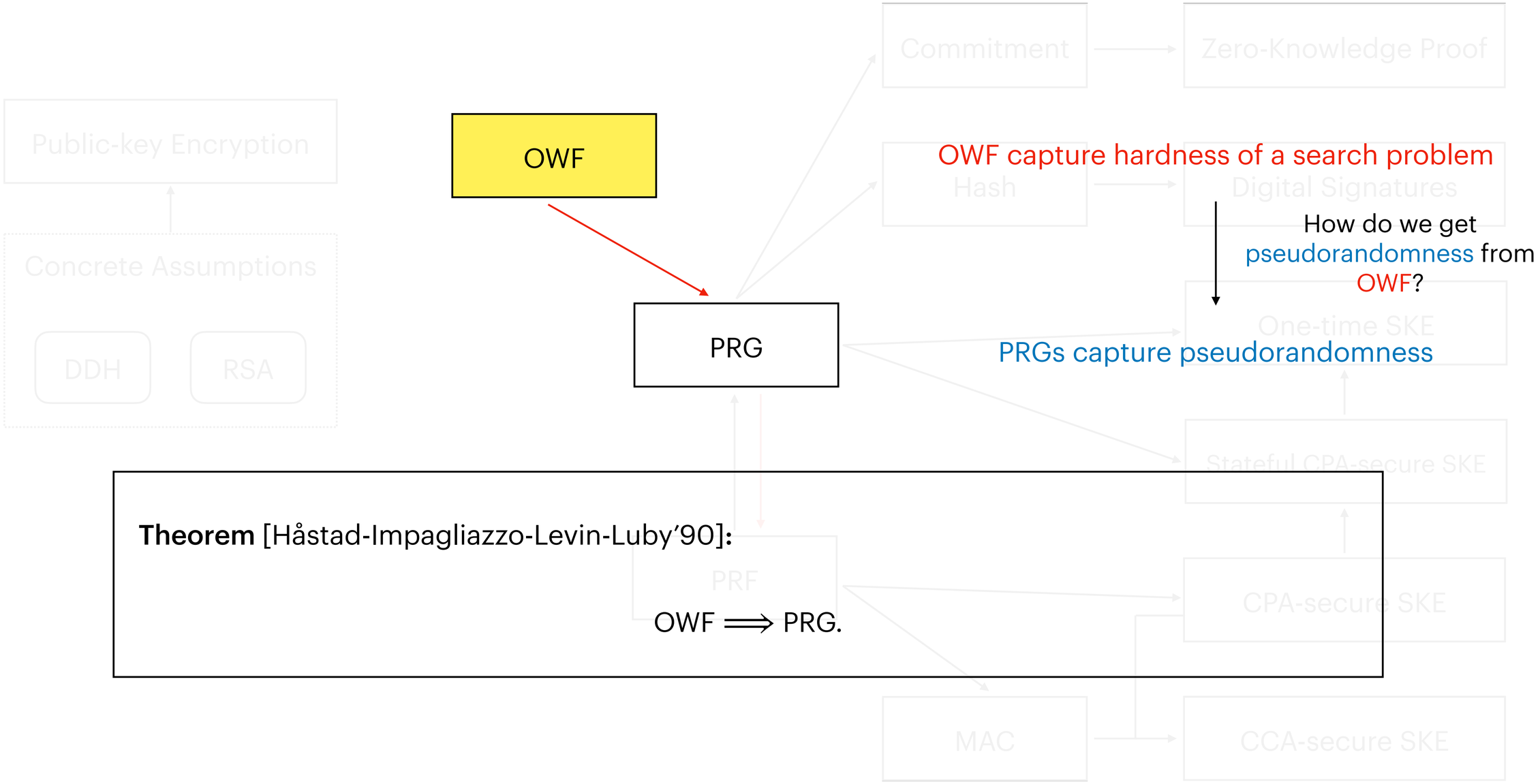

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Is there any non-trivial (non-identity) function of x, even 1 bit, that OWFs hide?

Hard-Core Predicate

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  - Intuitively, learning $\mathsf{hc}(x)$, even given $f(x)$, is as hard as inverting f.

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Hard-core predicates help get pseudorandomness from OWF.

PRGs from OWF

Theorem [Håstad-Impagliazzo-Levin-Luby'90]**:**

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Thus, if $G(x)$ is distinguishable from a uniformly random string, then it must be because of appending $\mathsf{hc}(x)$. But $\mathsf{hc}(x)$ is hard to predict even given $f(x)$.