Zero-Knowledge Proofs

601.442/642 Modern Cryptography

24th March 2026

Recap

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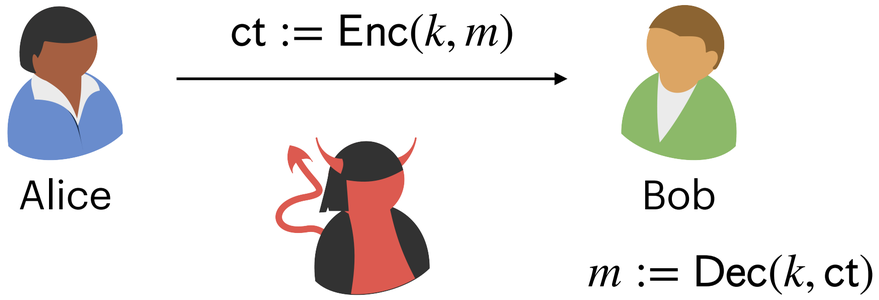
Encryption



Private communication in the presence of an eavesdropper.

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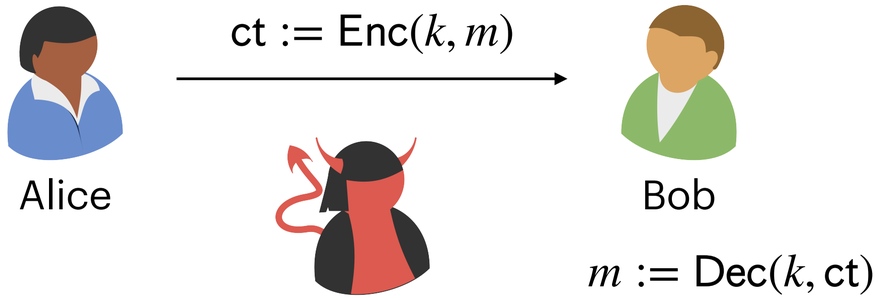

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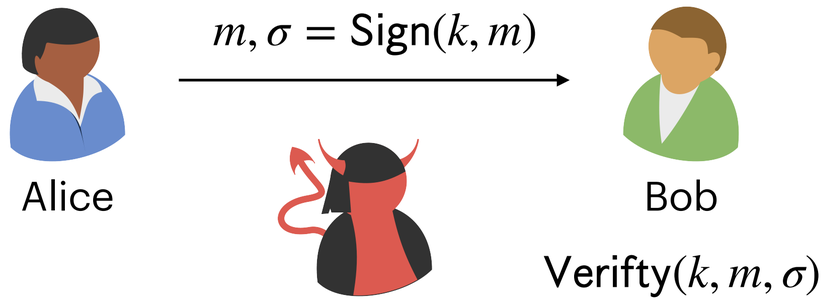

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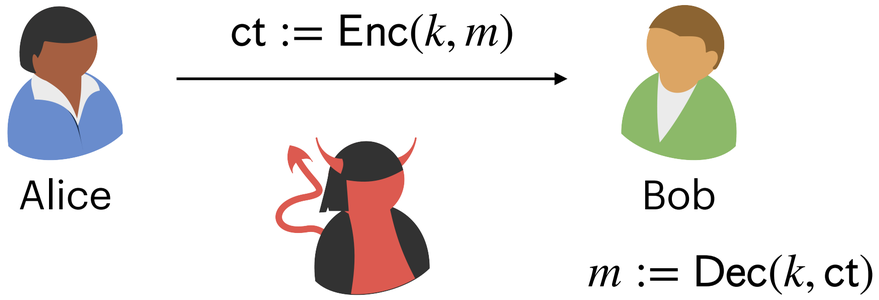

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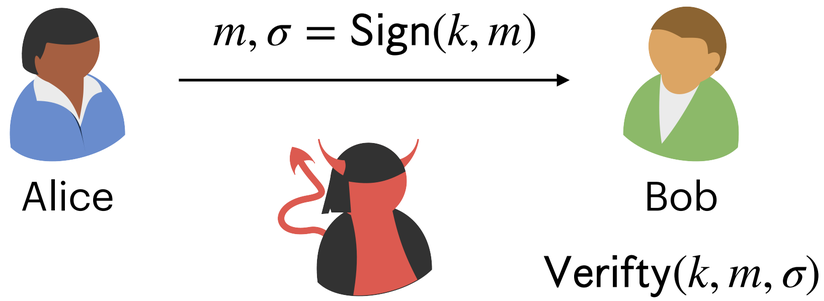

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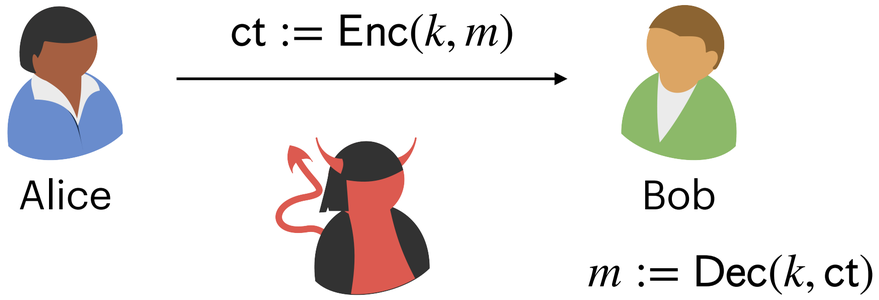

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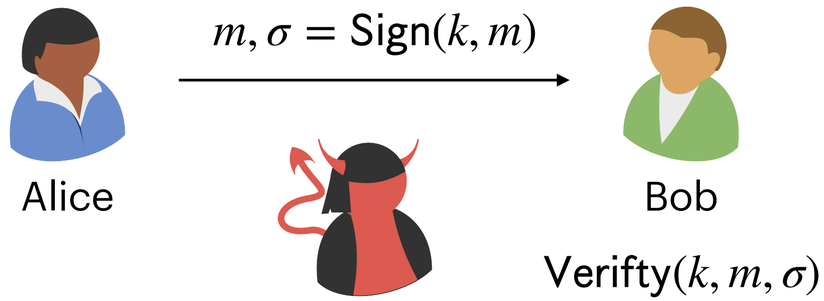
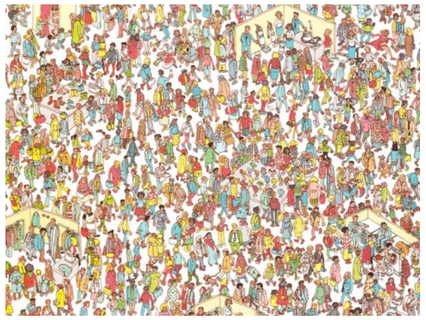


Authentic communication in the presence of an impersonator.

What if Alice and Bob do not trust each other?

What do they even want to achieve?

Example: Where’s Waldo?



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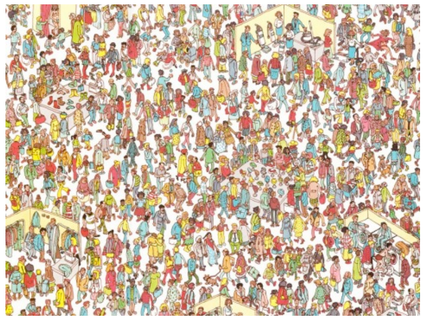

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“I found Waldo.”

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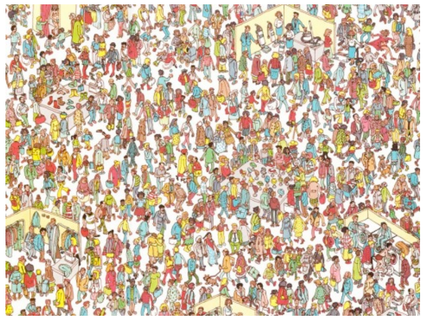

Alice

“I found Waldo.”

Bob

“That was way too fast, I don’t believe you.
Show me where?”

Example: Authentication

Alice

Bob

Example: Authentication

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Bob

“I’m Alice, can I access the database?”

Example: Authentication

Alice

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“I’m Alice, can I access the database?”

“I don’t trust you. Give me Alice’s password.”

Example: Age Verification

Alice

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“I’m older than 21, can I get a beer?”

Example: Age Verification

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“I’m older than 21, can I get a beer?”

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“I don’t believe you. Give me your ID.”

A Problem of Trust and Knowledge

- **Proofs:** Alice wants to convince an untrusting Bob of something.
  - She found Waldo.
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  - Waldo's location
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Zero-Knowledge Proofs: History

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Paper rejected three times! Accepted the fourth time in 1985.

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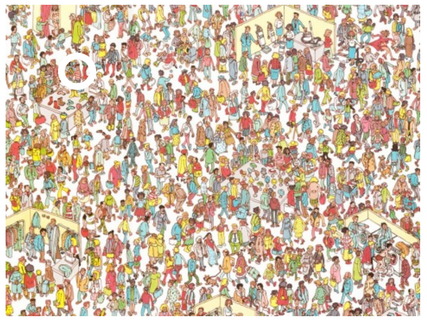

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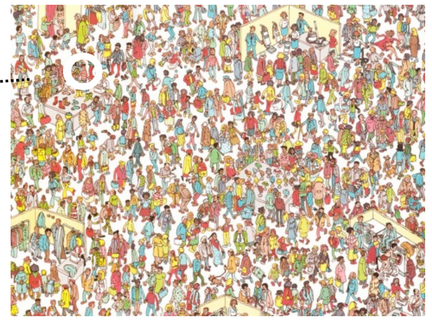

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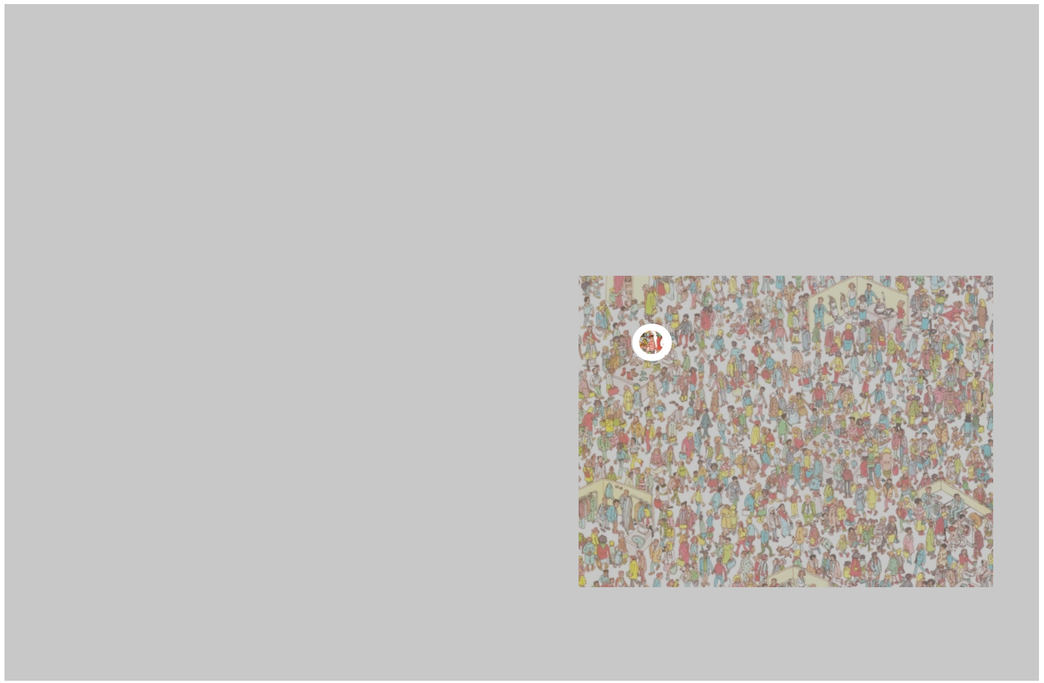

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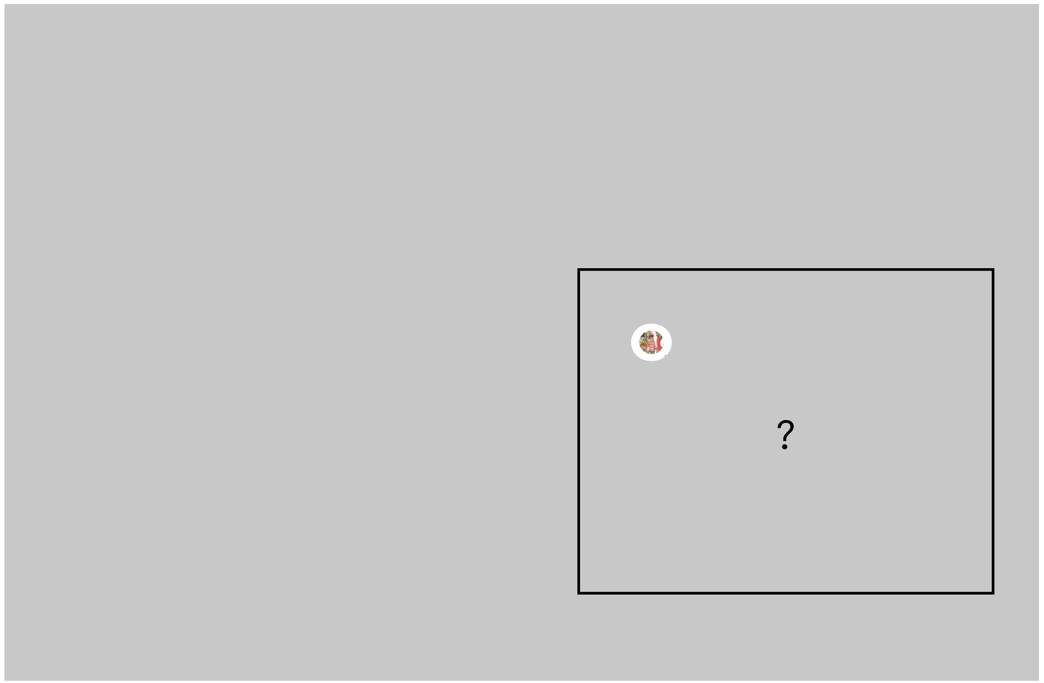

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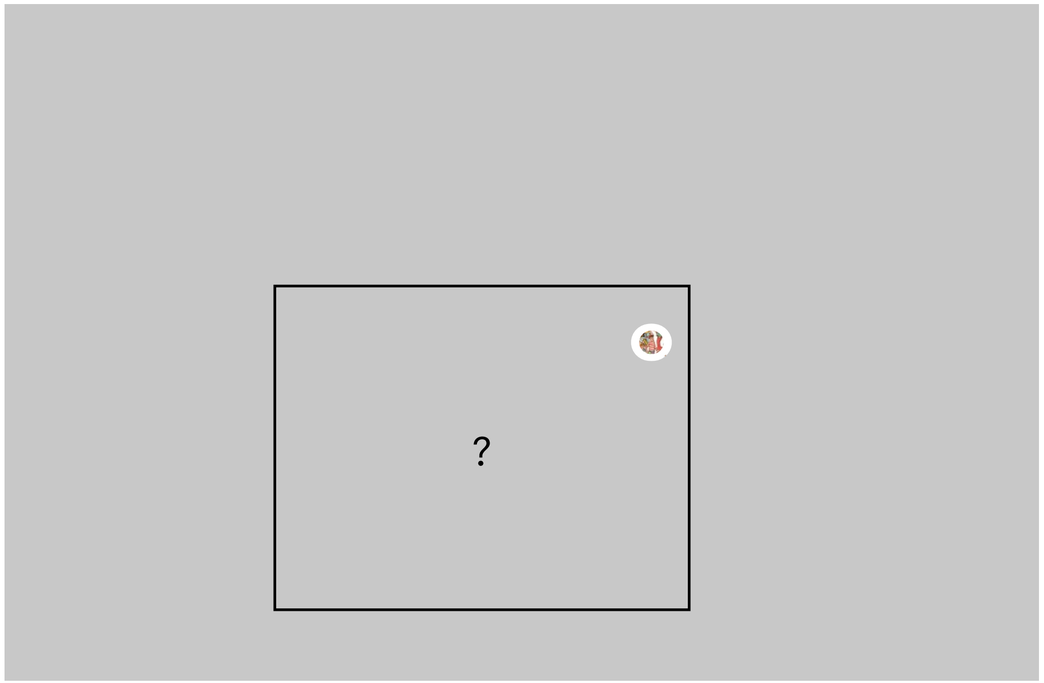

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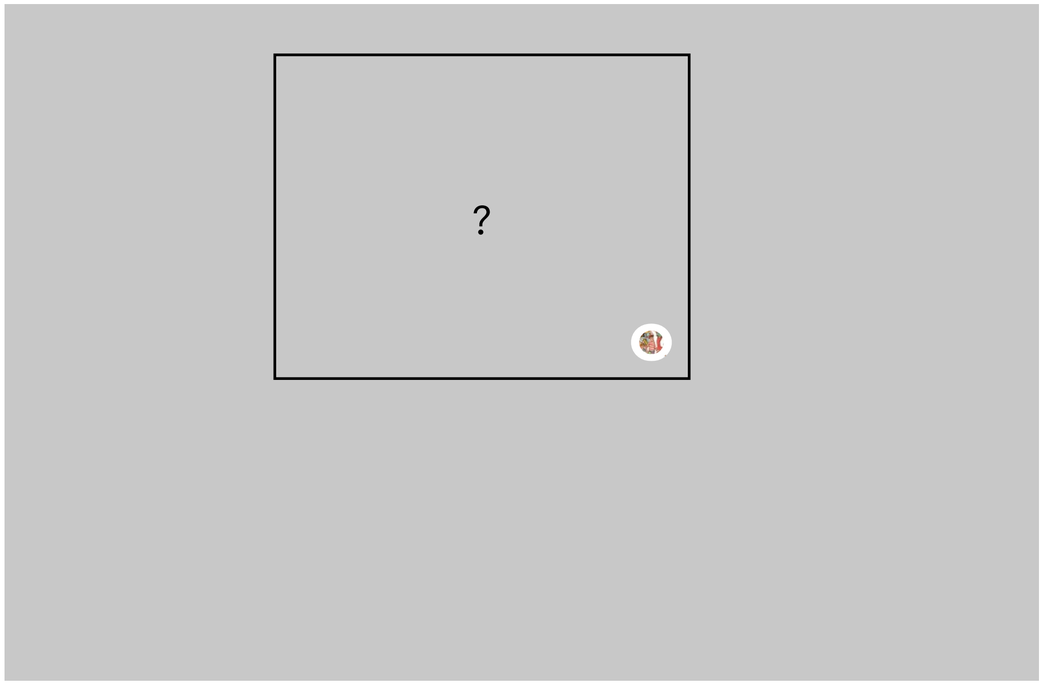

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How to define zero-knowledge?

Proofs

Statement x : $(P \vee Q) \wedge (P \vee R) \implies P \vee (Q \wedge R)$

Proof π :

1	$(P \vee Q) \wedge (P \vee R)$	
2	$P \vee Q$	∧E 1
3	$P \vee R$	∧E 1
4	$\quad P$	
5	$\quad P \vee (Q \wedge R)$	∨I 4
6	$\quad Q$	
7	$\quad\quad P$	
8	$\quad\quad P \vee (Q \wedge R)$	∨I 7
9	$\quad\quad R$	
10	$\quad\quad Q \wedge R$	∧I 6, 9
11	$\quad\quad P \vee (Q \wedge R)$	∨I 10
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Verifier should accept the proof if x is true.

Verifier should reject the proof if x is false.

Proof System

Helpful to think about a prover trying to convince a verifier that the statement is true.

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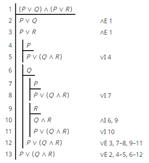

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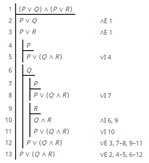

Verifier

"I don't trust you."

A proof is needed to convince a verifier that does not trust the prover.

Generalizing to Decision Problems

Statement x

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"$x \in L$"

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Language: Set of true statements.

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Think written proof vs seminar/talk

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  - **Completeness:** P can be used to convince V to accept a true statement.
  - **Soundness:** V will always reject a false statement.
  - **Efficient Verification**: V must be polynomial time in the length of the statement.

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Prover and verifier can use randomness.

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- Proof system for languages beyond what is captured by non-interactive proofs.

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Prover

Glass A

Glass B

Verifier

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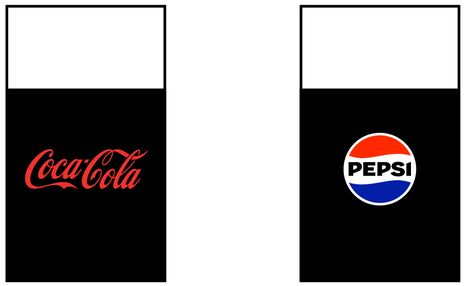

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How do we convince the Verifier?

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Else, it wins with probability 1/2. Repeat to amplify soundness.

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    - Multiple provers [Babai-Fortnow-Lund]: MIP = NEXP.

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    - Multiple provers [Babai-Fortnow-Lund]: MIP = NEXP.
- Achieving privacy guarantee for the prover.

The Power of Interaction and Randomness

- Proof system for languages beyond what is captured by non-interactive proofs.
  - Languages in NP have a non-interactive proof.
  - Interactive proofs can prove statements in languages not known to be in NP.
    - Single prover [Shamir]: IP = PSPACE.
    - Multiple provers [Babai-Fortnow-Lund]: MIP = NEXP.
- Achieving privacy guarantee for the prover.
  - **Zero knowledge:** Verifier learns nothing from the proof beyond the validity of the statement.