Zero-Knowledge Proofs IV

601.442/642 Modern Cryptography

2nd April 2026

Recap: Commitment Schemes

Commitment Scheme

Let $\ell := \ell(\lambda)$ be a polynomial. A PPT algorithm Com is a commitment scheme for ℓ-bit strings if it satisfies the following properties.

- **Hiding:** For every $v_0, v_1 \in \{0,1\}^\ell$, we have

$$\{\mathsf{Com}(v_0\ ;\ r) : r \xleftarrow{\$} \{0,1\}^\lambda\}$$

$$\overset{c}{\approx}\ \{\mathsf{Com}(v_1\ ;\ r) : r \xleftarrow{\$} \{0,1\}^\lambda\}.$$

- **Binding:** For all $v_0, v_1 \in \{0,1\}^\ell$ and $r_0, r_1 \in \{0,1\}^\lambda$, such that $v_0 \neq v_1$, we have

$$\mathsf{Com}(v_0\ ;\ r_0) \neq \mathsf{Com}(v_1\ ;\ r_1).$$

Recap: Bit Commitments from OWP

Theorem: If one-way permutations exist, then there exists a commitment scheme for 1-bit messages.

Proof:

Let $f : \{0,1\}^\lambda \to \{0,1\}^\lambda$ be a OWP and let $\mathsf{hc} : \{0,1\}^\lambda \to \{0,1\}$ be the hardcore predicate for f.

$$\mathsf{Com}(b\ ;\ r) = \Big(f(r),\ \mathsf{hc}(r) \oplus b\Big)$$

Can also be constructed from OWFs, but requires a bit more effort.

Lemma: A bit commitment implies a commitment scheme for polynomial length bit strings.

Simply commit to each bit separately.

ZKP for Graph 3-Coloring

Statement: $G = (V, E)$ where $|V| = n = \text{poly}(\lambda)$.

Prover's NP witness: Color assignment $\phi : V \to \{1, 2, 3\}$.

ZKP for Graph 3-Coloring

Repeat the following procedure $n|E|$ times.

- $P \to V$: P samples $\psi \xleftarrow{\$} \text{Perm}_3$ uniformly at random. For each vertex $v_i \in V$ it computes $\text{color}_i := \psi\left(\phi(v_i)\right)$ and

  $$c_i := \text{Com}(\text{color}_i)$$

  and sends $(c_1, \ldots, c_n)$ to V.

- $V \to P$: V samples a random edge (i, j) and sends it to P.

- $P \to V$: P opens c_i and c_j to reveal color_i and color_j.

- V: If the openings are valid and $\text{color}_i \neq \text{color}_j$, then V continues to the next iteration. Else, it rejects.

If V does not reject in any iteration, it accepts the proof.

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- Sample $(i', j') \xleftarrow{\$} E$ and colors $\text{color}_{i'}, \text{color}_{j'} \xleftarrow{\$} \{1, 2, 3\}$ such that $\text{color}_{i'} \neq \text{color}_{j'}$. Set every other $\text{color}_i = 1$.

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- If $(i, j) = (i', j')$, then output $(c_1, \ldots, c_n)$, (i, j), and the opening for $c_{i'}$ and $c_{j'}$ as the verifier's view. Else, restart the above process.

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- If simulation has not succeeded after $n|E|$ attempts, then output $\perp$.

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ZKP for Graph 3-Coloring

Goal: Show that the view of $\hat{V}$ in the protocol is indistinguishable from simulated view.

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ZKP for Graph 3-Coloring

Hybrid 0

- $\psi \xleftarrow{\$} \mathsf{Perm}_3$
- $\forall i, \mathsf{color}_i := \psi\left(\phi(v_i)\right)$
- $\forall i, c_i := \mathsf{Com}(\mathsf{color}_i)$
- $(i, j) \leftarrow \widehat{V}(c_1, \ldots, c_n)$

Output

$(c_1, \ldots, c_n)$ (i, j) Openings for c_i and c_j

This hybrid is $\widehat{V}$'s view in a real execution of the protocol.

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Generating this distribution requires ϕ.

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Hybrid 1

If $(i, j) \neq (i', j')$

At most $n|E|$ times

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This hybrid is similar to the real world prover, but keeps trying until it guesses $\widehat{V}$'s response.

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Claim: Hybrid 0 $\overset{s}{\approx}$ Hybrid 1.

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Claim: Hybrid 0 $\overset{s}{\approx}$ Hybrid 1.

The hybrids are identical conditioned on $(i,j) = (i',j')$ since (i',j') is sampled uniformly at random.

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The hybrids are identical conditioned on $(i,j) = (i',j')$ since (i',j') is sampled uniformly at random.

Therefore, we need to argue that the output is not $\perp$ with overwhelming probability.

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(i,j) is independent of (i',j') since $(c_1, \ldots, c_n)$ is independent of (i',j').

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If $(i, j) \neq (i', j')$

At most $n|E|$ times

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(i, j) is independent of (i', j') since $(c_1, \ldots, c_n)$ is independent of (i', j').

Therefore, probability of not terminating after $n|E|$ attempts is

$$\left(1 - \frac{1}{|E|}\right)^{n|E|} \approx e^{-n} = \text{negl}(\lambda).$$

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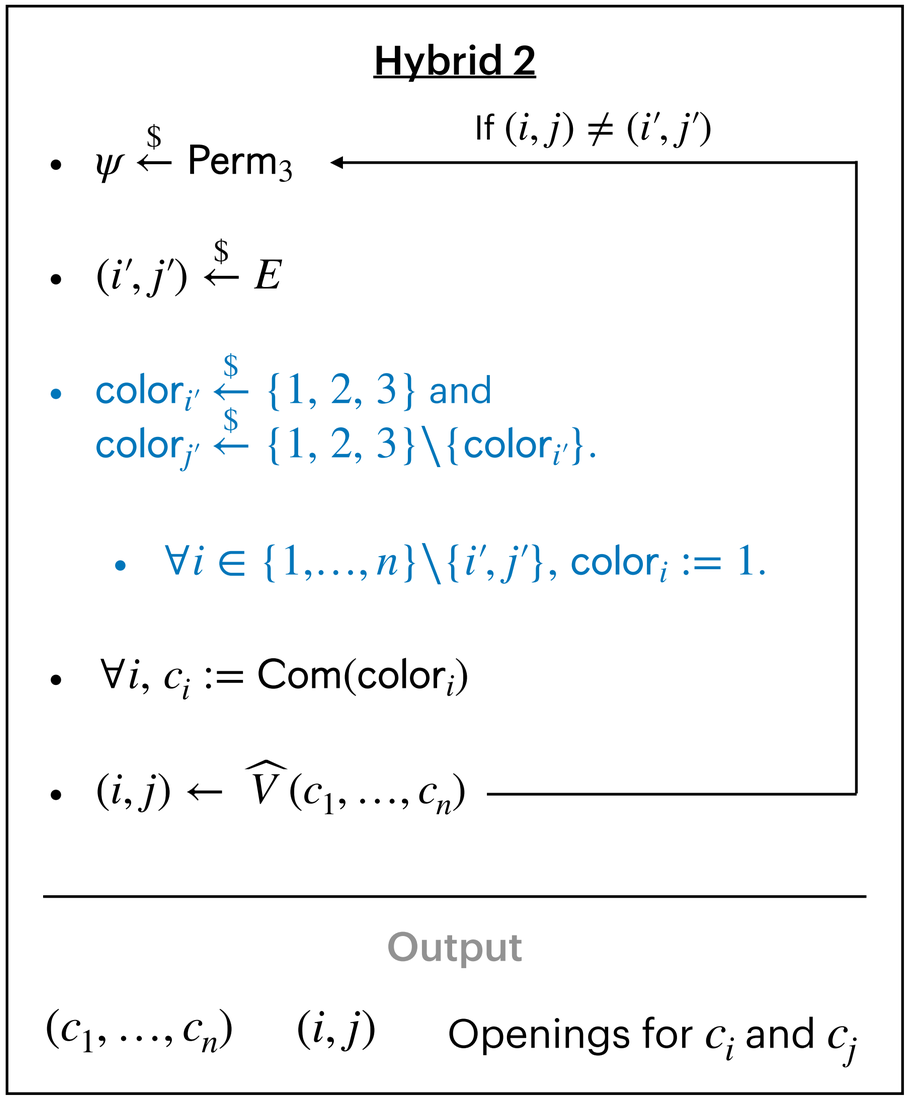
This hybrid still requires ϕ.

ZKP for Graph 3-Coloring



This hybrid is the simulator S.

ZKP for Graph 3-Coloring

Hybrid 2

If $(i,j) \neq (i',j')$

- $\psi \xleftarrow{\$} \mathsf{Perm}_3$
- $(i',j') \xleftarrow{\$} E$
- $\mathsf{color}_{i'} \xleftarrow{\$} \{1,2,3\}$ and $\mathsf{color}_{j'} \xleftarrow{\$} \{1,2,3\}\backslash\{\mathsf{color}_{i'}\}$.
  - $\forall i \in \{1,\ldots,n\}\backslash\{i',j'\}$, $\mathsf{color}_i := 1$.
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Claim: Hybrid 1 $\overset{c}{\approx}$ Hybrid 2.

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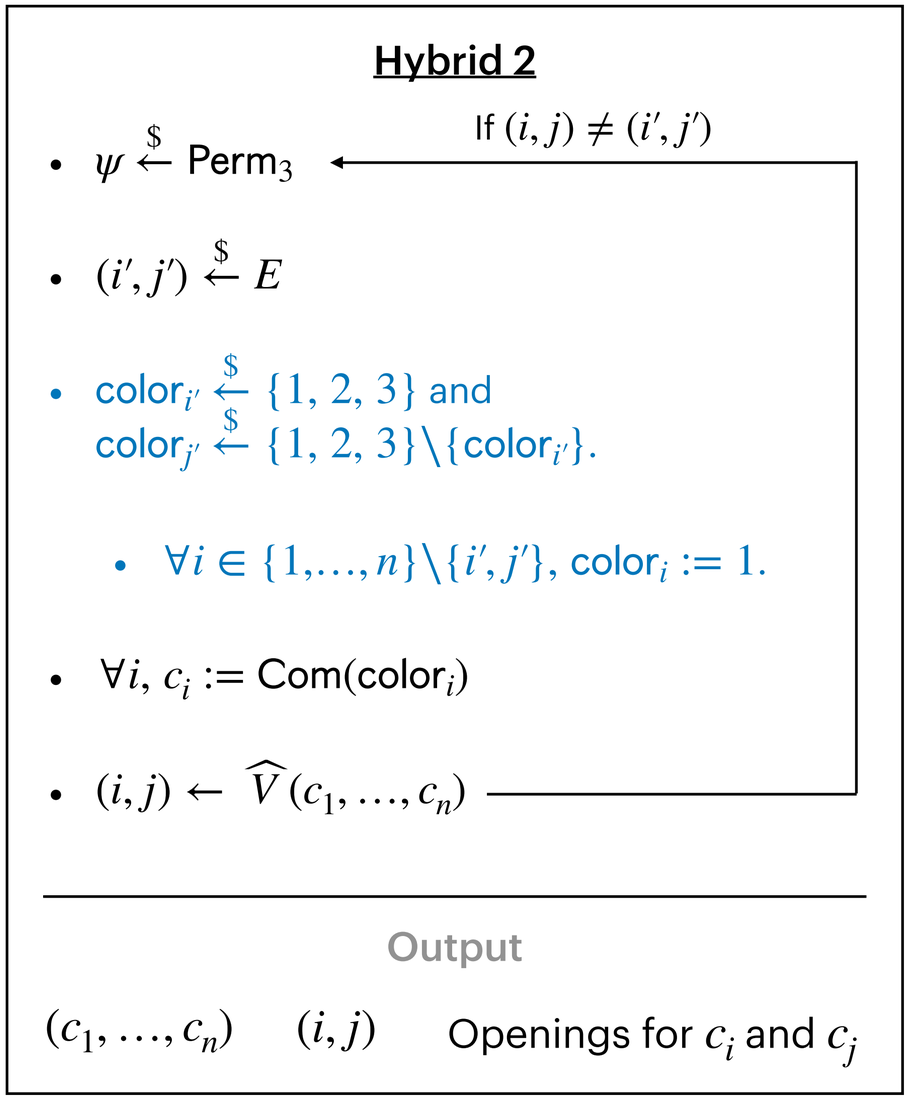

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Claim: Hybrid 1 $\overset{c}{\approx}$ Hybrid 2.

Distribution of $\{\text{color}_i\}_i$ now depends on (i',j').

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Distribution of $\{\mathsf{color}_i\}_i$ now depends on (i',j').

However, by hiding property of commitment scheme, $\{c_i\}_i$ is computationally indistinguishable across hybrids.

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$\Longrightarrow$ (i,j) is computationally indistinguishable across hybrids.

Probability that $(i,j) = (i',j')$ is negligibly close across hybrids.

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This hybrid no longer requires ϕ.

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View in Real Execution

- $\psi \xleftarrow{\$} \text{Perm}_3$
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Output

$(c_1, \ldots, c_n)$ (i, j) Openings for c_i and c_j

$\overset{c}{\approx}$

Simulated View

If $(i, j) \neq (i', j')$

- $\psi \xleftarrow{\$} \text{Perm}_3$
- $(i', j') \xleftarrow{\$} E$
- $\text{color}_{i'} \xleftarrow{\$} \{1, 2, 3\}$ and $\text{color}_{j'} \xleftarrow{\$} \{1, 2, 3\} \backslash \{\text{color}_{i'}\}$.
  - $\forall i \in \{1, \ldots, n\} \backslash \{i', j'\}, \text{color}_i := 1.$
- $\forall i, c_i := \text{Com}(\text{color}_i)$
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Round Complexity of ZKP

Statement: $G = (V, E)$ where $|V| = n = \text{poly}(\lambda)$.

Prover's NP witness: Color assignment $\phi : V \to \{1, 2, 3\}$.

ZKP for Graph 3-Coloring

Repeat the following procedure $n|E|$ times.

- $P \to V$: P samples $\psi \xleftarrow{\$} \text{Perm}_3$ uniformly at random. For each vertex $v_i \in V$ it computes $\text{color}_i := \psi\left(\phi(v_i)\right)$ and

  $$c_i := \text{Com}(\text{color}_i)$$

  and sends $(c_1, \ldots, c_n)$ to V.

- $V \to P$: V samples a random edge (i, j) and sends it to P.

- $P \to V$: P opens c_i and c_j to reveal color_i and color_j.

- V: If the openings are valid and $\text{color}_i \neq \text{color}_j$, then V continues to the next iteration. Else, it rejects.

If V does not reject in any iteration, it accepts the proof.

Requires $3 \cdot n|E|$ rounds.

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Can we run all repetitions in parallel to get a 3-round ZKP?

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Can we run all repetitions in parallel to get a 3-round ZKP?

Might not preserve ZK in general!

Theorem [Goldreich-Krawczyk'90]**:** There exist ZKPs whose parallel repetition is not zero knowledge.

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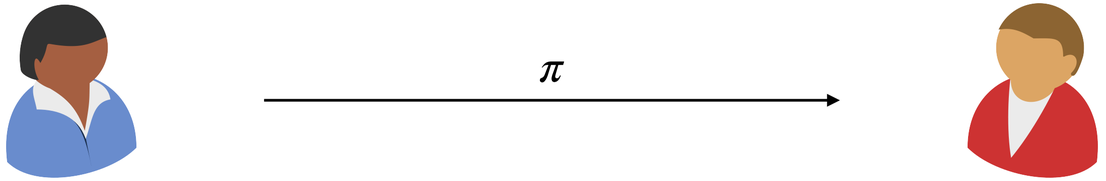
Can we construct a one-message ZKP?

Non-Interactive Zero-Knowledge Proofs

$x \in L$

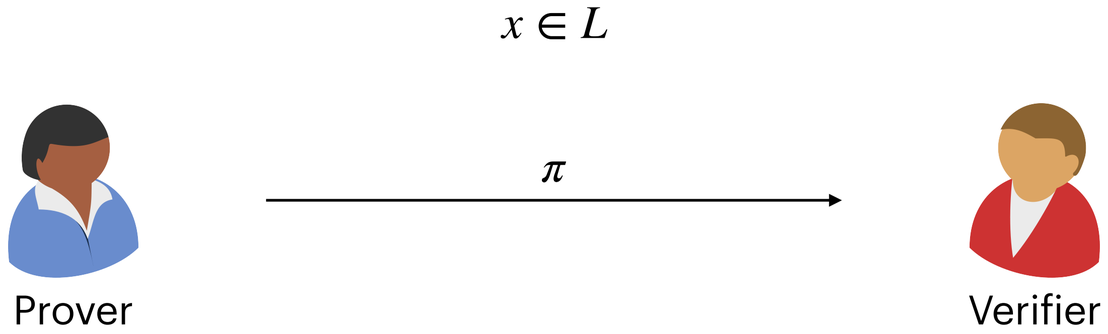


Prover

Verifier

We want completeness, soundness and zero-knowledge.

Non-Interactive Zero-Knowledge Proofs



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Interaction is necessary for ZK!

Theorem: If a language L has a one-message ZKP, then $L \in$ BPP.

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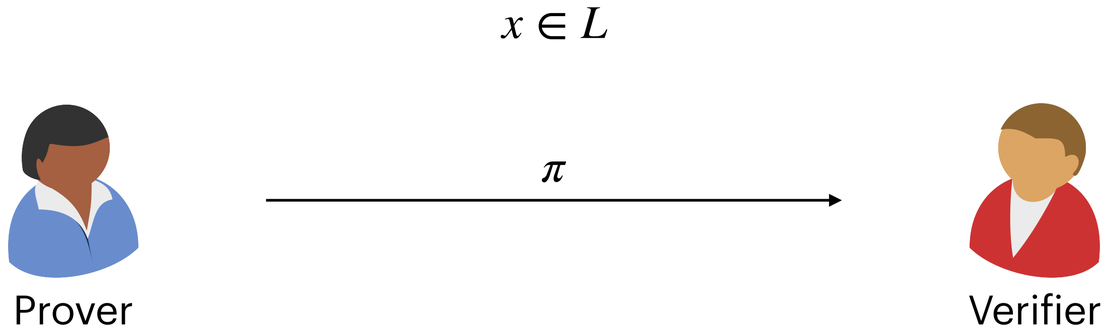

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Only languages that can be decided by a PPT algorithm have one-message ZKP.

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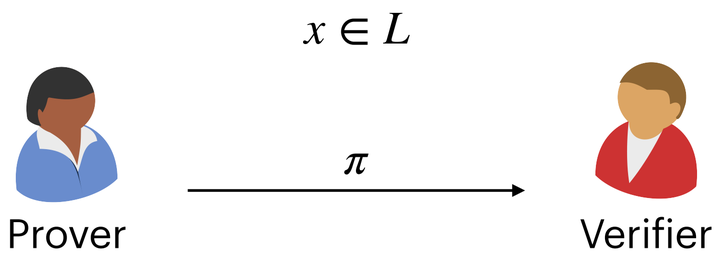

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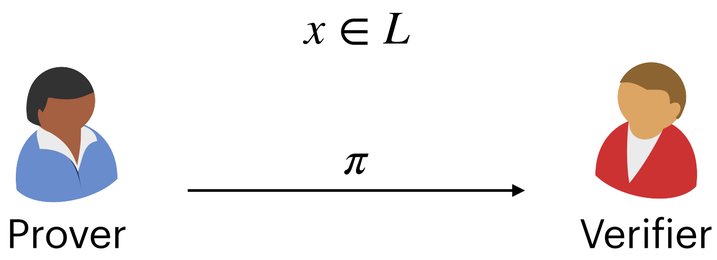

Verifier accepts π when $x \in L$ (completeness).

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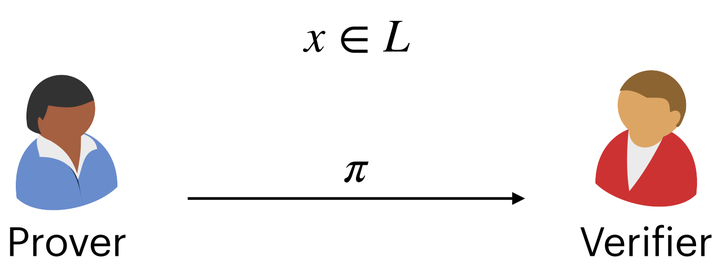

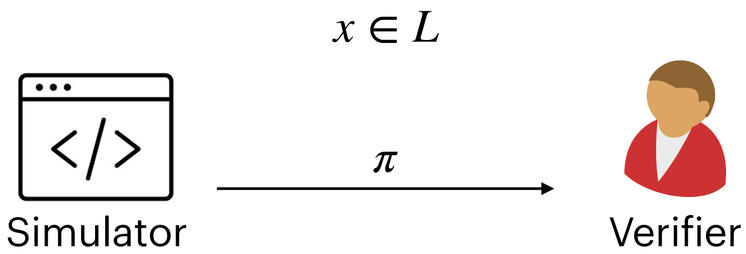

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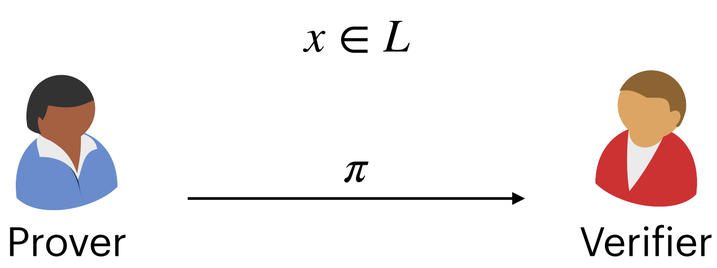

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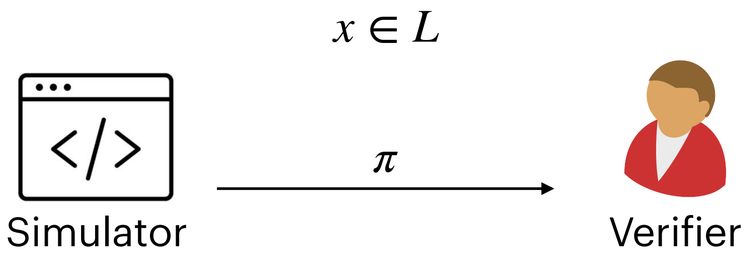

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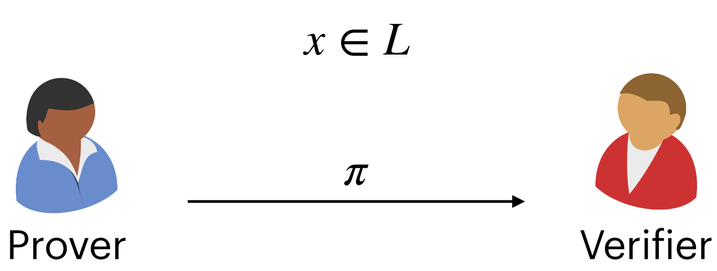

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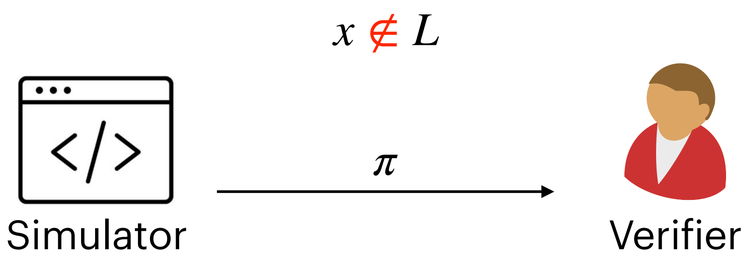

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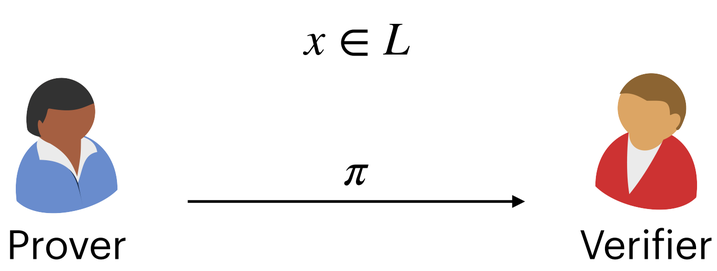

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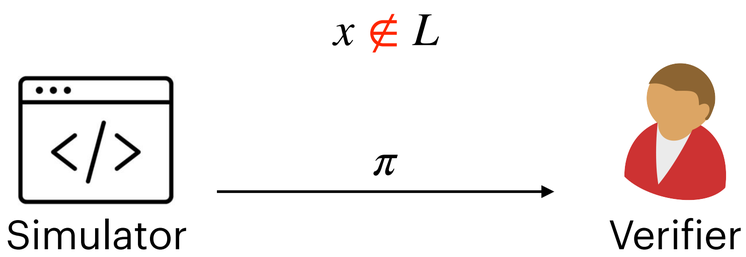

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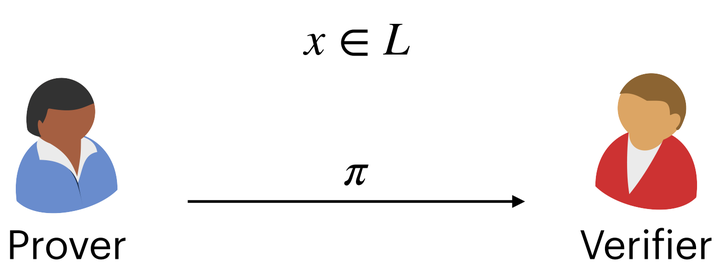

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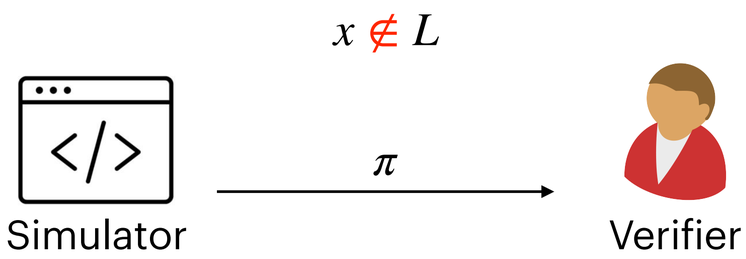

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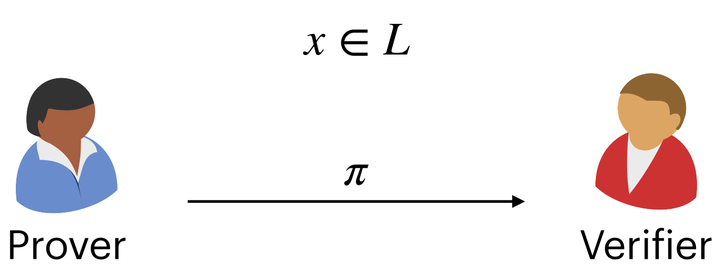

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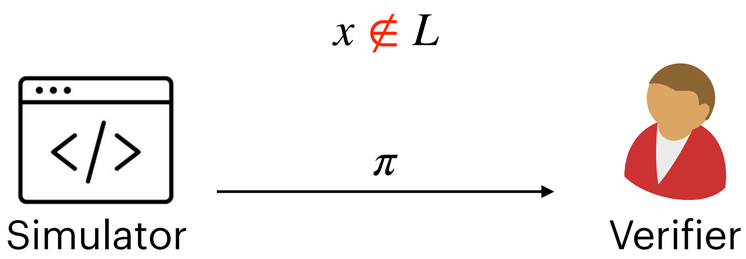

Claim: Verifier accepts π.

Why?

S and V are PPT algorithms.

If $L \notin$ BPP then neither S nor V can decide if $x \in L$.

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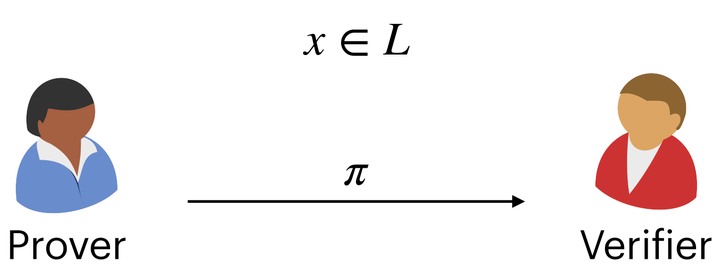

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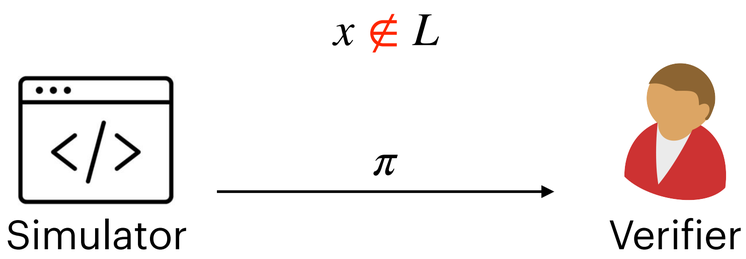

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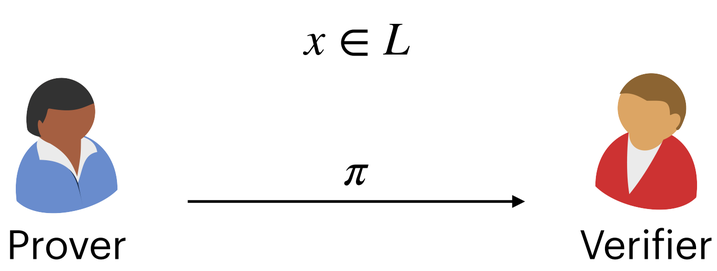

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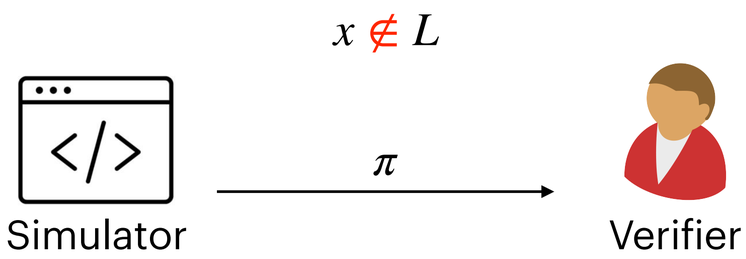

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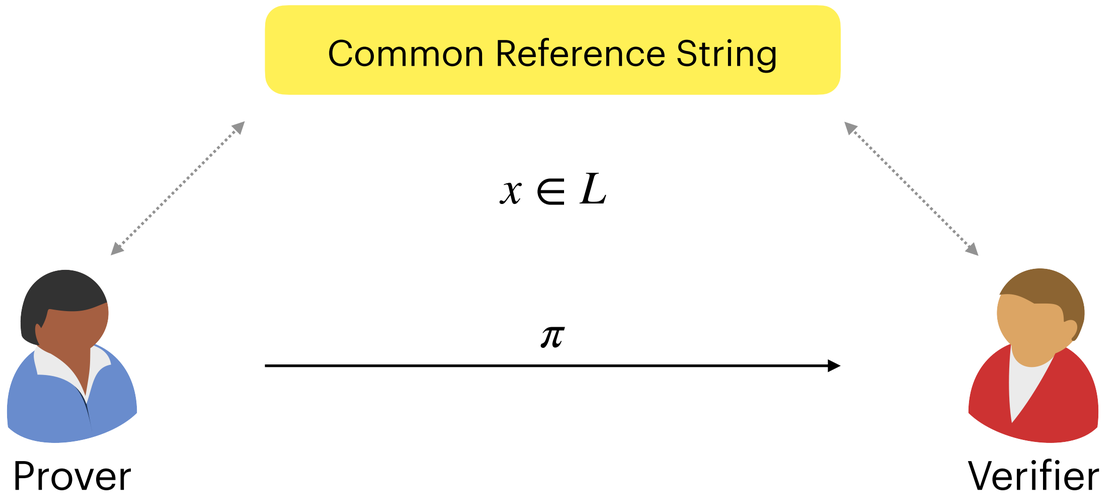
The proof system is not sound!

Non-Interactive Zero-Knowledge Proofs

Define NIZKs in the common reference string model.



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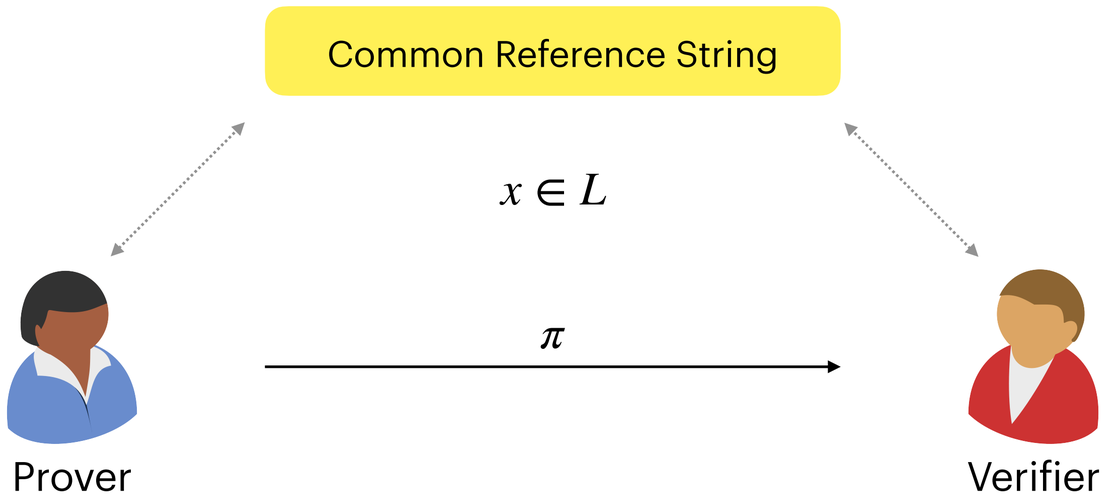

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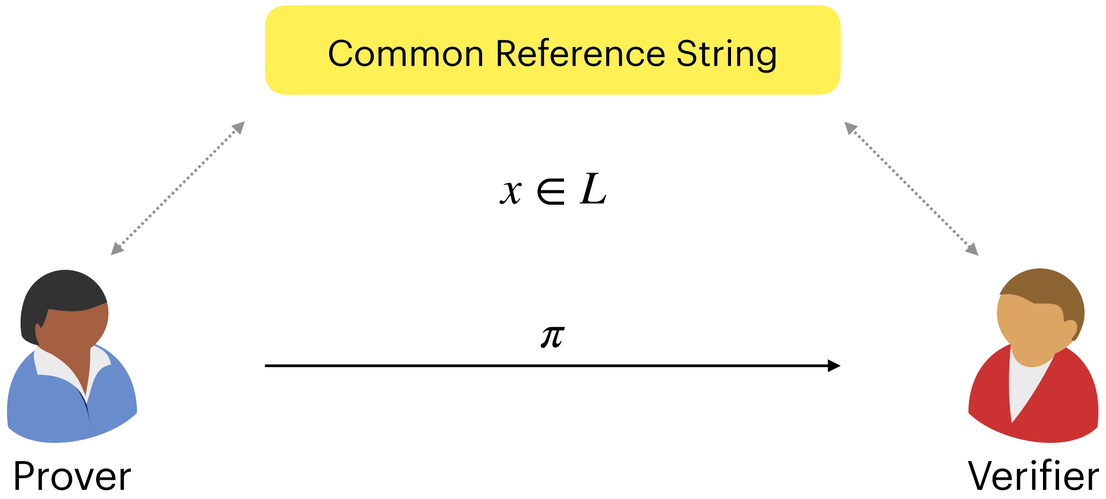

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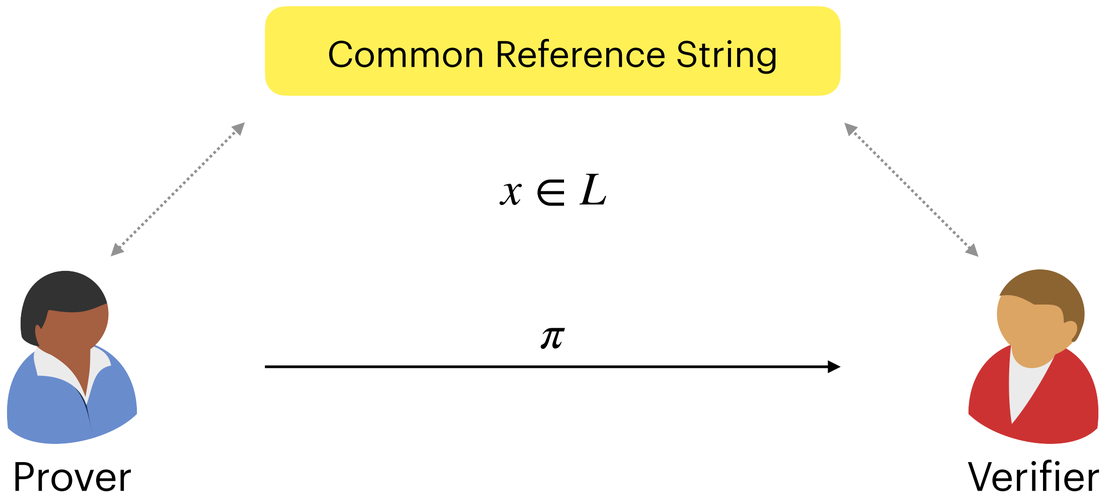

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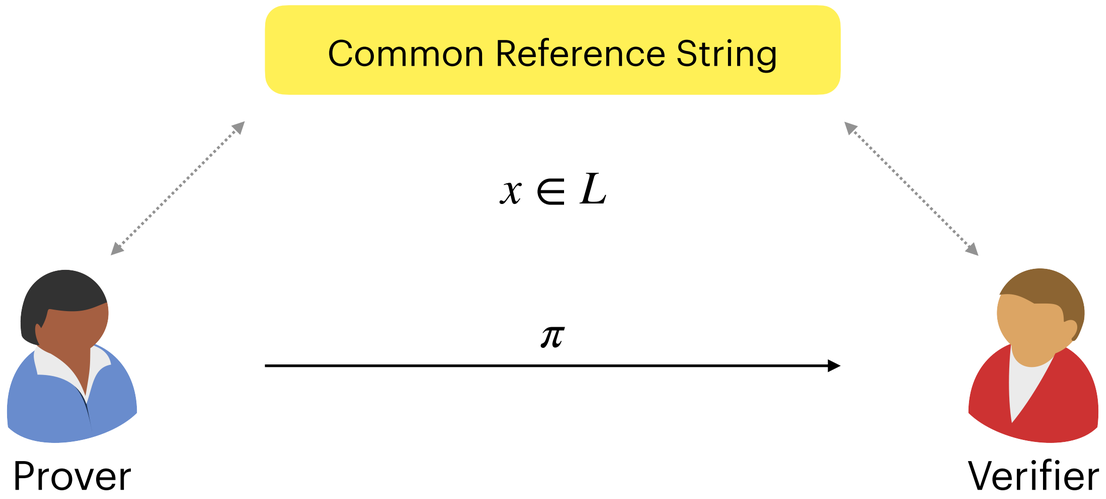

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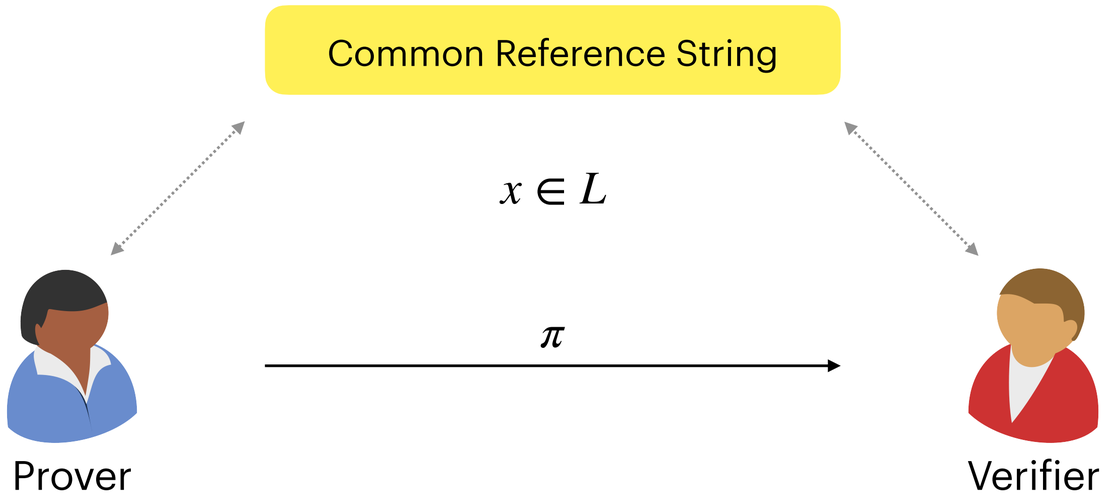

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- **Fiat-Shamir Transform:** Converting any public-coin ZKP into a NIZK in the random oracle model.
  - In a public-coin ZKP, all verifier messages are uniformly random.

The Fiat-Shamir Transform

Applied to ZKP for Graph 3-Coloring

Statement $x : G = (V, E)$

Prover

Verifier

Let $\phi : V \to \{1, 2, 3\}$ be a 3-coloring of G.

$\psi \xleftarrow{\$} \mathsf{Perm}_3$.

$\forall i \in \{1, \ldots, n\}$:

$\mathsf{color}_i := \psi\left(\phi(v_i)\right)$

$c_i := \mathsf{Com}\left(\mathsf{color}_i\right)$

$c_1, \ldots, c_n$ →

Sample a random edge $(i, j) \xleftarrow{\$} E$.

← (i, j)

Openings for c_i, c_j →

Check if openings are valid and if $\mathsf{color}_i \neq \mathsf{color}_j$.

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If the hash function is like a "random function", it outputs uniformly random (i, j) based on the prover's first message.

The Fiat-Shamir Transform

Applied to ZKP for Graph 3-Coloring

Statement $x : G = (V, E)$

Prover

Verifier

This ZKP is public-coin!

The verifier's samples a random edge after the prover has committed to a coloring.

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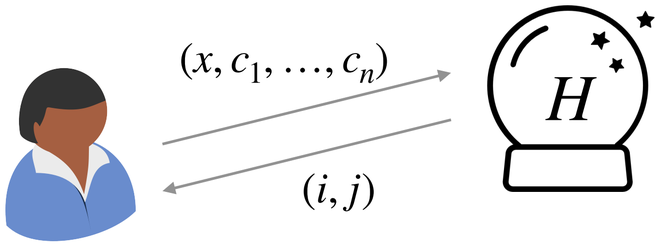

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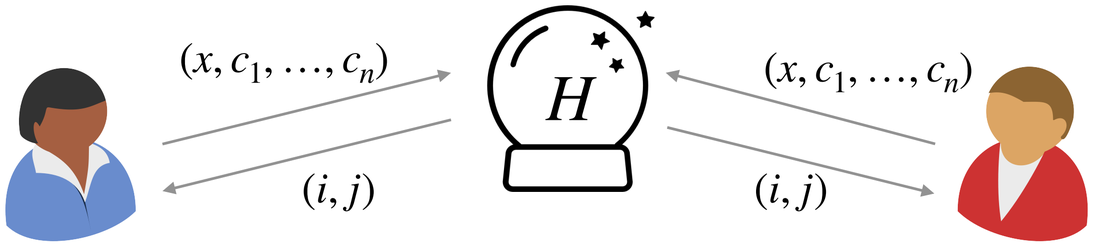

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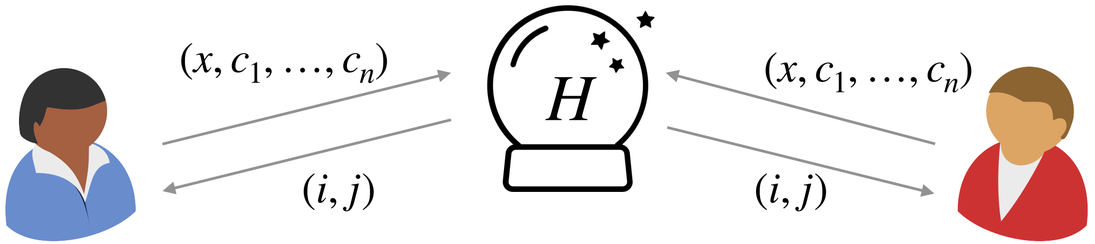

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The random oracle is like an honest verifier. It always returns uniformly random edges.

The Fiat-Shamir Transform

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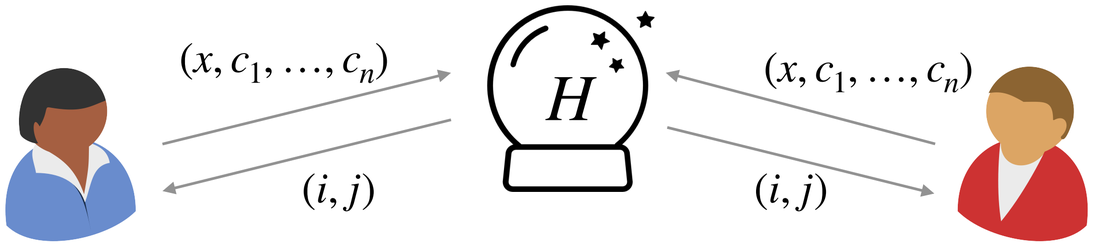

- Is this protocol sound?

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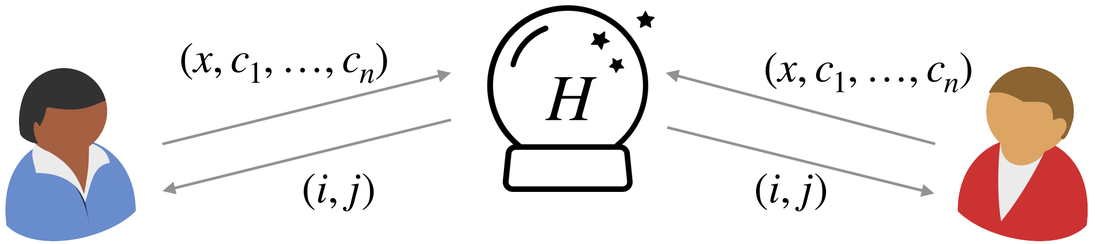

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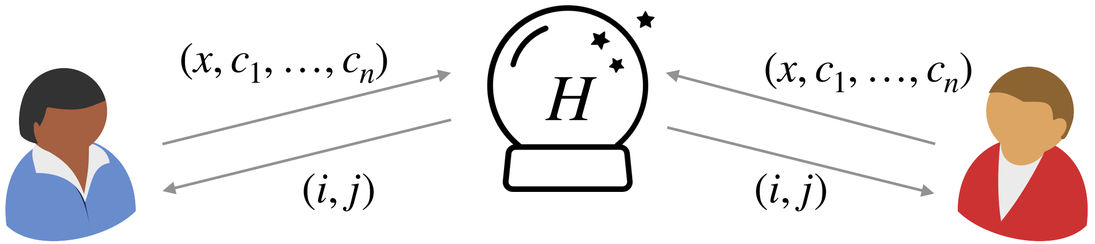

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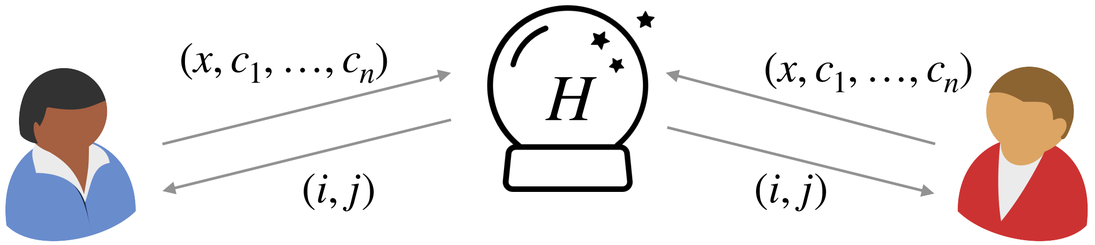

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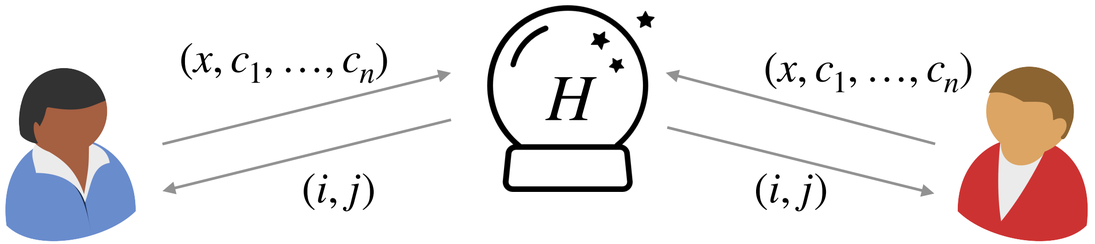

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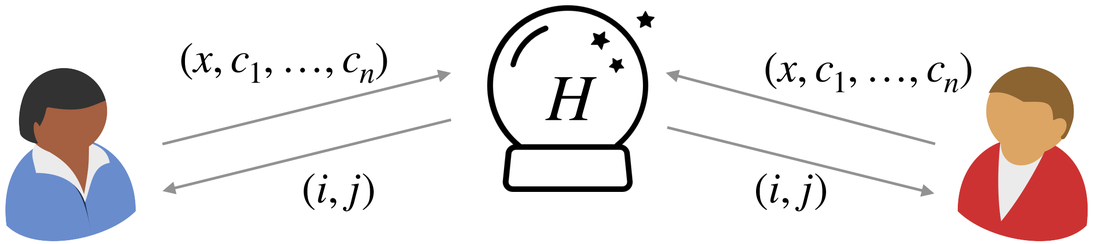

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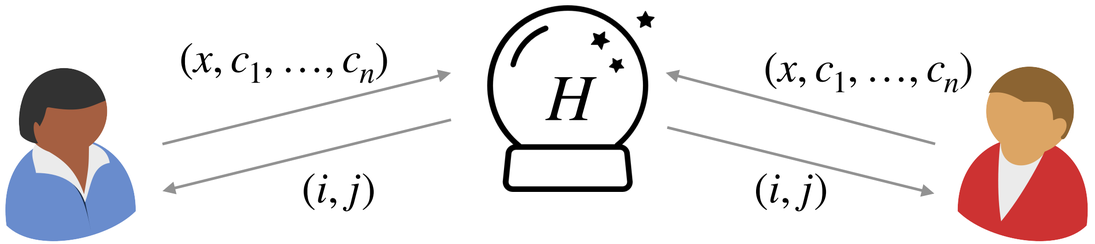

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Solution: Use parallel repetition!

The Fiat-Shamir Transform

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Statement $x : G = (V, E)$

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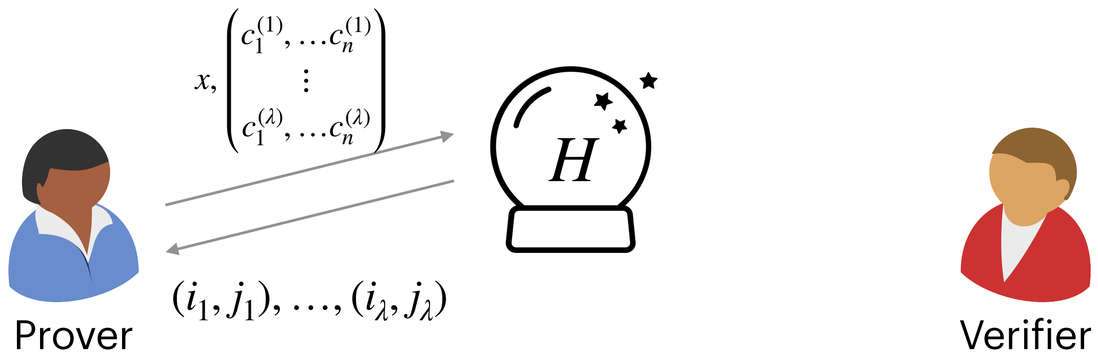

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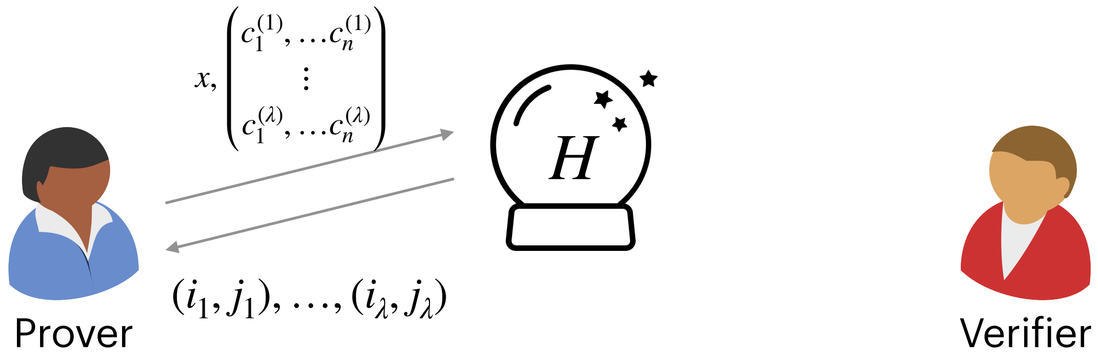

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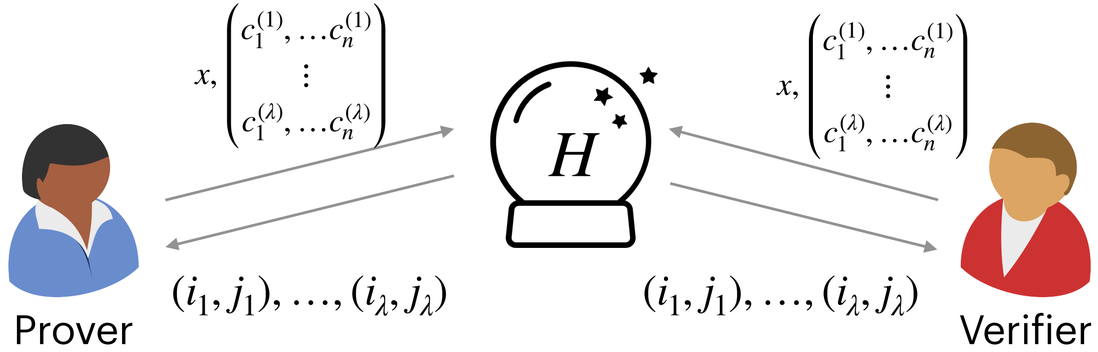

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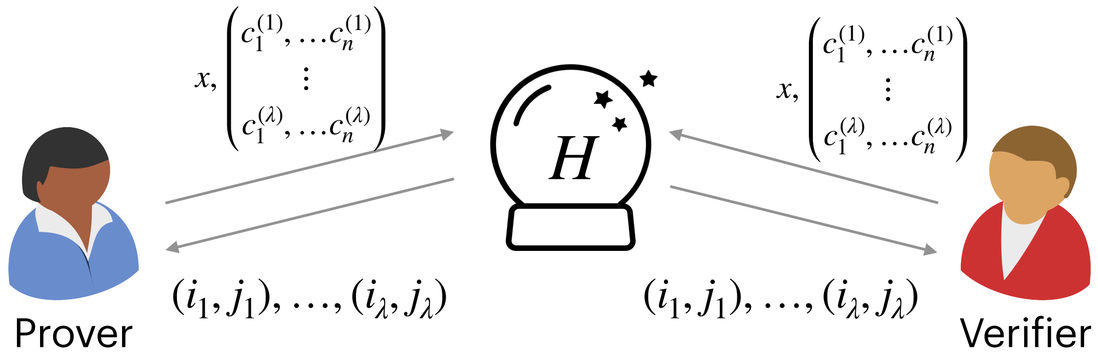

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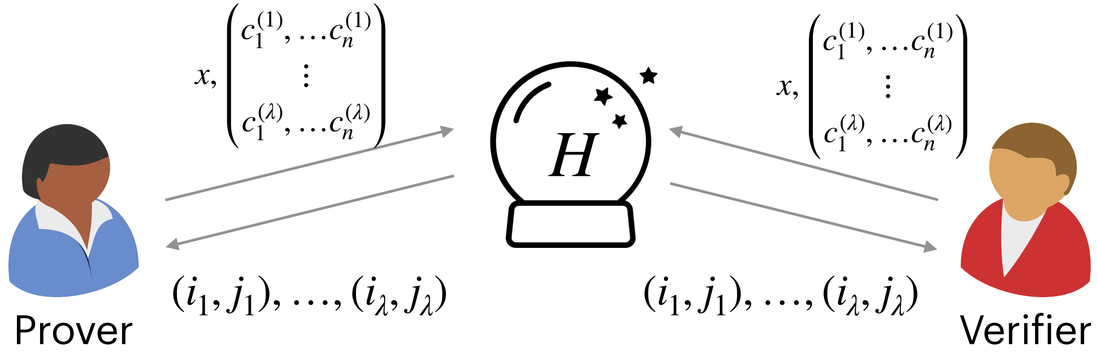

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Soundness:

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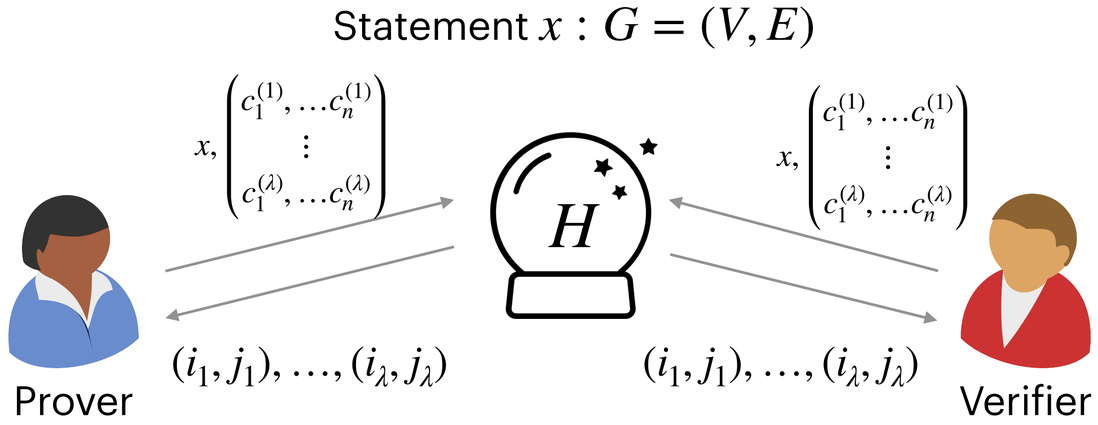

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Soundness: Malicious prover would have to guess all challenge edges to cheat.

Check if all openings are valid and if $\mathsf{color}_i^{(k)} \neq \mathsf{color}_j^{(k)}$.

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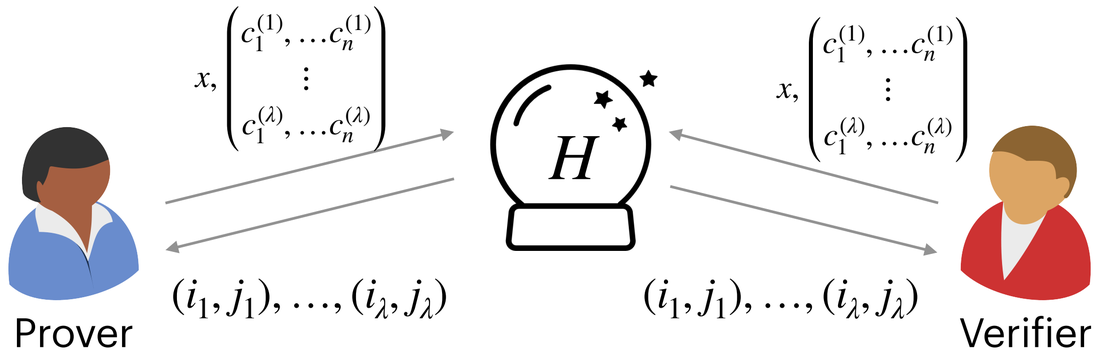

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Zero Knowledge:

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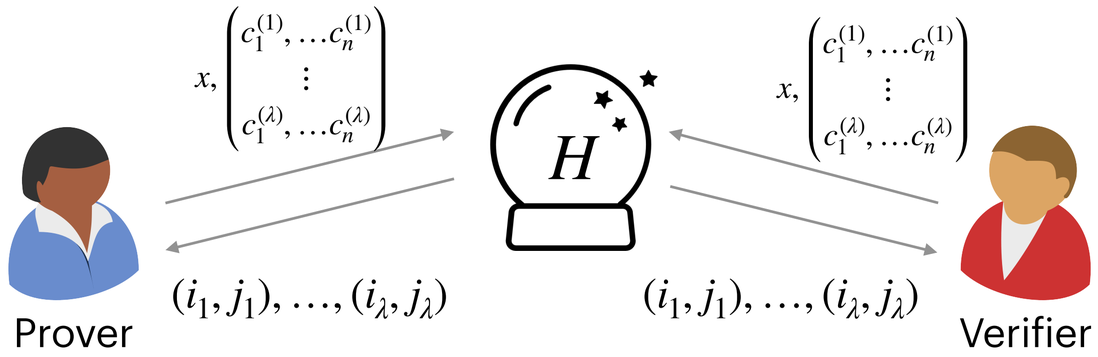

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Zero Knowledge: Prover never interacts with the malicious verifier and the random oracle is like an “honest verifier”.

In the proof, the simulator can program the random oracle.

Zero-Knowledge Proof of Knowledge

Until now we have considered decision problems:

Either the statement is true ($x \in L$) or false ($x \notin L$).

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Wants to prove that she has the secret key s s.t. h $= g^s$.

Prover

$$x = (\mathbb{G}, g, p, h)$$

Verifier

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"There exists s such that $h = g^s$."

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$x = (\mathbb{G}, g, p, h)$

Verifier

"There exists s such that $h = g^s$."

- This statement is always true since g generates $\mathbb{G}$.

Zero-Knowledge Proof of Knowledge

Until now we have considered decision problems:

Either the statement is true ($x \in L$) or false ($x \notin L$).

(Either there exists a 3-coloring or there does not exist a 3-coloring.)

Wants to prove that she has the secret key s s.t. $h = g^s$.

Prover

$$x = (\mathbb{G}, g, p, h)$$

Verifier

"There exists s such that $h = g^s$."

- This statement is always true since g generates $\mathbb{G}$.
- This is not an interesting decision problem.

Zero-Knowledge Proof of Knowledge

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Verifier

"There exists s such that $h = g^s$."

- This statement is always true since g generates $\mathbb{G}$.
- This is not an interesting decision problem.
- It is a search problem.

Zero-Knowledge Proof of Knowledge

Until now we have considered decision problems:

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Wants to prove that she has the secret key s s.t. h $= g^s$.

Prover

$x = (\mathbb{G}, g, p, h)$

Verifier

"I know s such that $h = g^s$."

Zero-Knowledge Proof of Knowledge

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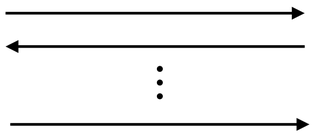

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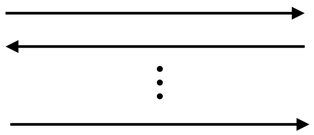

"I'm convinced you know the solution."

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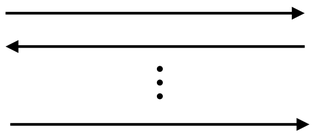

"I'm convinced you know the solution."

Completeness: The verifier accepts when it interacts with the prover run with input s.

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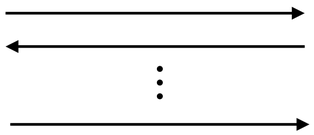

"I'm convinced you know the solution."

Completeness: The verifier accepts when it interacts with the prover run with input s.

Zero-Knowledge: For every (possibly malicious) verifier, there is a simulator whose output is indistinguishable from the verifier's view in an interaction with the prover.

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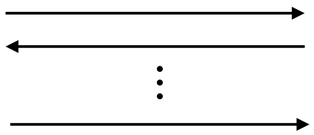

"I'm convinced you know the solution."

Completeness: The verifier accepts when it interacts with the prover run with input s.

Soundness: What does it mean to say a machine *knows* something?

Zero-Knowledge: For every (possibly malicious) verifier, there is a simulator whose output is indistinguishable from the verifier's view in an interaction with the prover.

Zero-Knowledge Proof of Knowledge

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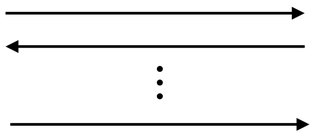

"I'm convinced you know the solution."

Completeness: The verifier accepts when it interacts with the prover run with input s.

Soundness: If a (possibly malicious) prover convinces the verifier then there is an efficient algorithm E, called the extractor, that outputs the witness s.

Zero-Knowledge: For every (possibly malicious) verifier, there is a simulator whose output is indistinguishable from the verifier's view in an interaction with the prover.

Zero-Knowledge Proof of Knowledge

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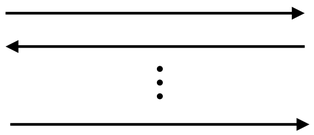


"I'm convinced you know the solution."

Completeness: The verifier accepts when it interacts with the prover run with input s.

Soundness: If a (possibly malicious) prover convinces the verifier then there is an efficient algorithm E, called the extractor, that outputs the witness s.

We will not formally define extractors in this course.

Zero-Knowledge: For every (possibly malicious) verifier, there is a simulator whose output is indistinguishable from the verifier's view in an interaction with the prover.

Zero-Knowledge Proof of Knowledge

Theorem [Goldreich-Micali-Wigderson'87]**:** Assuming OWFs exist, any language in NP has a zero-knowledge proof of knowledge.

Additional Slides

ZK Proof of Knowledge for Discrete Log

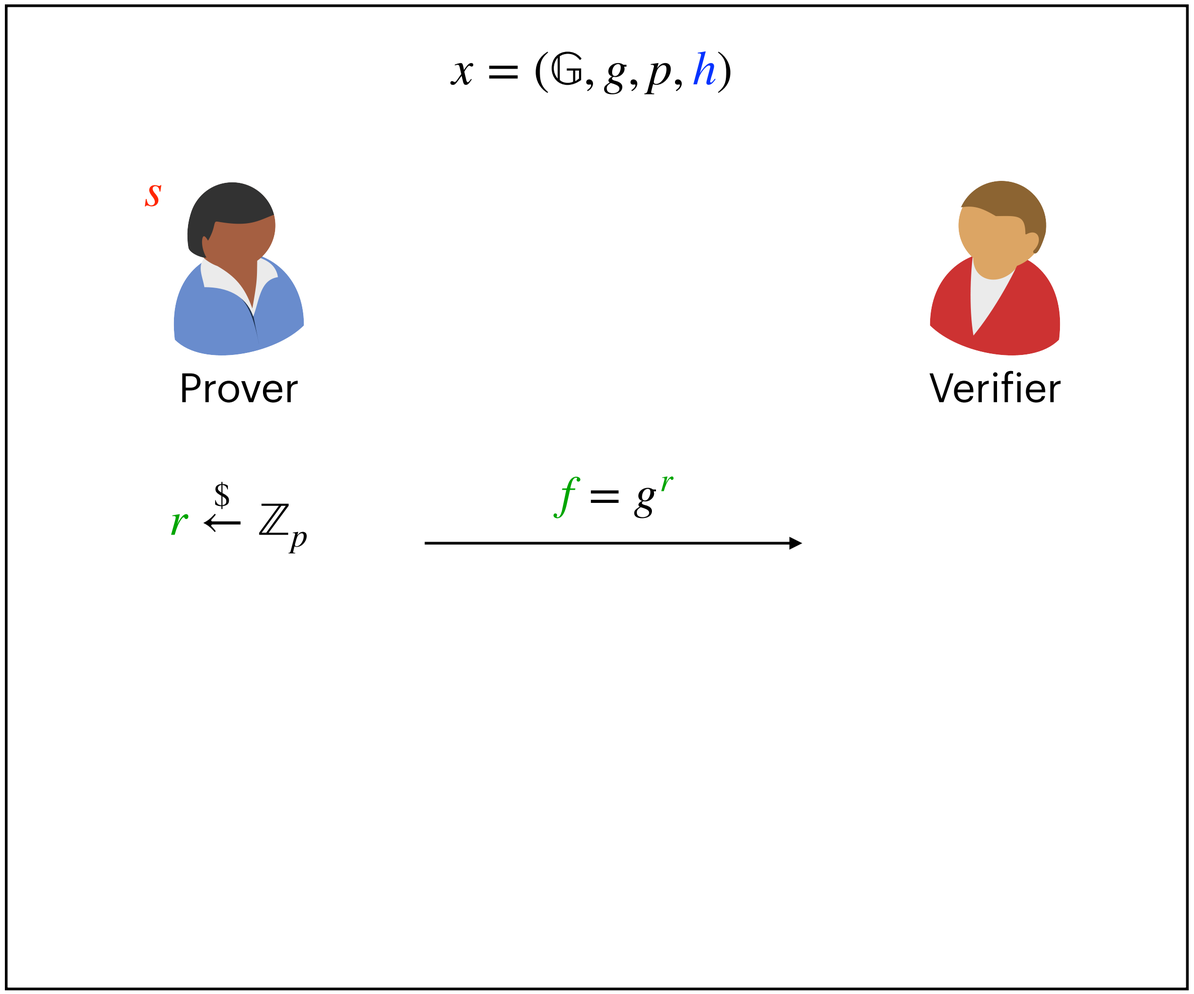
$x = (\mathbb{G}, g, p, h)$

s

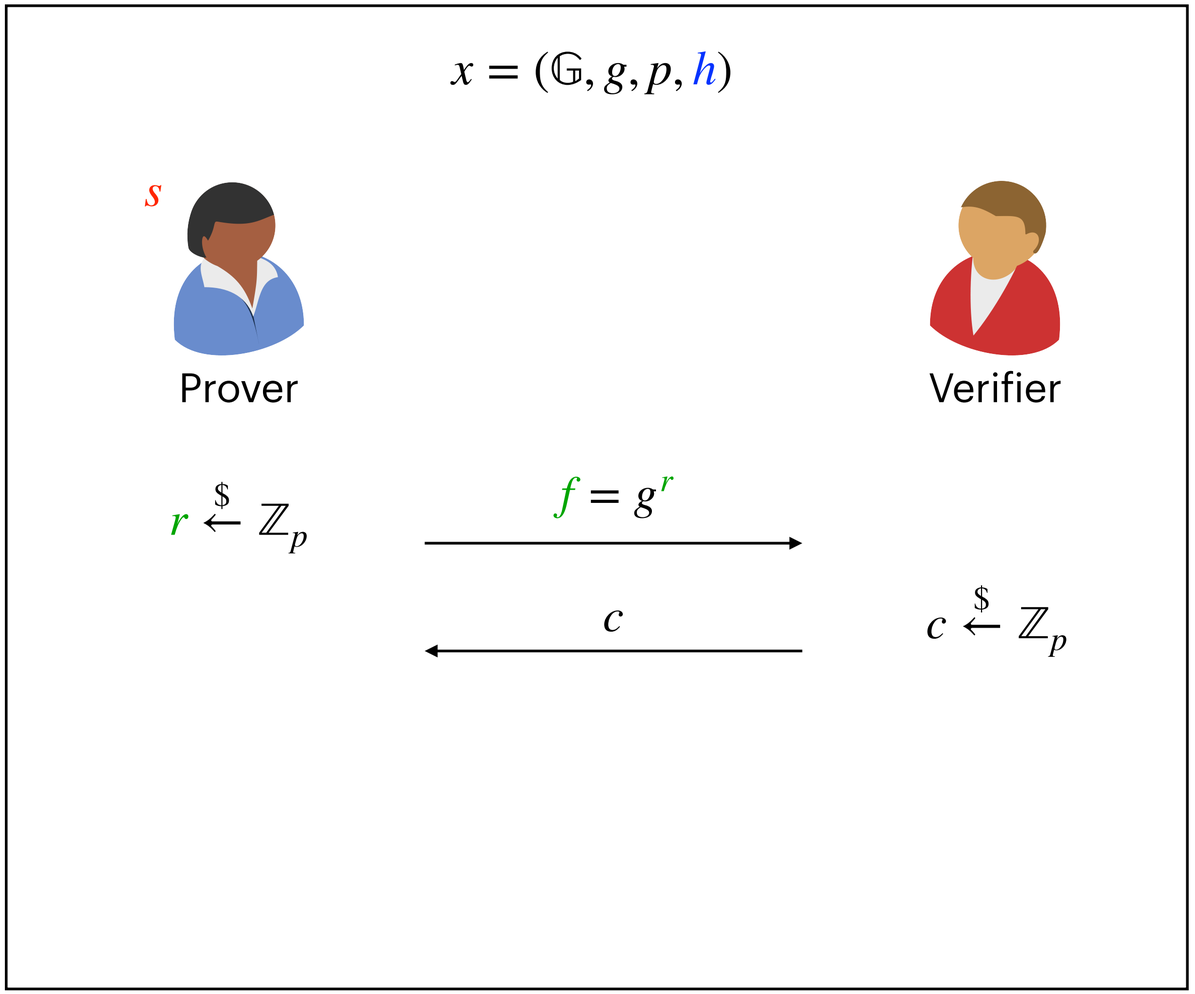
Prover

Verifier

ZK Proof of Knowledge for Discrete Log



ZK Proof of Knowledge for Discrete Log



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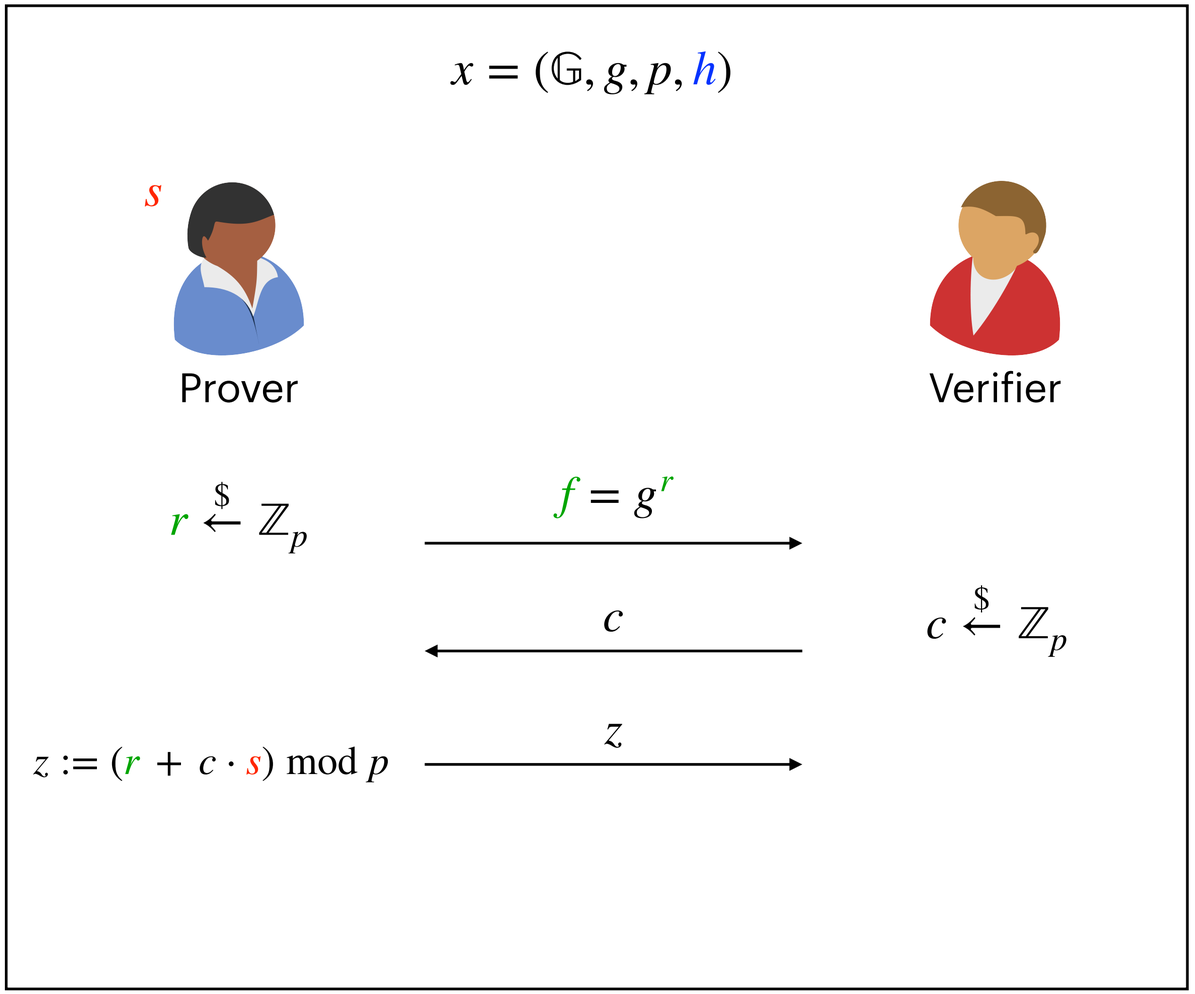

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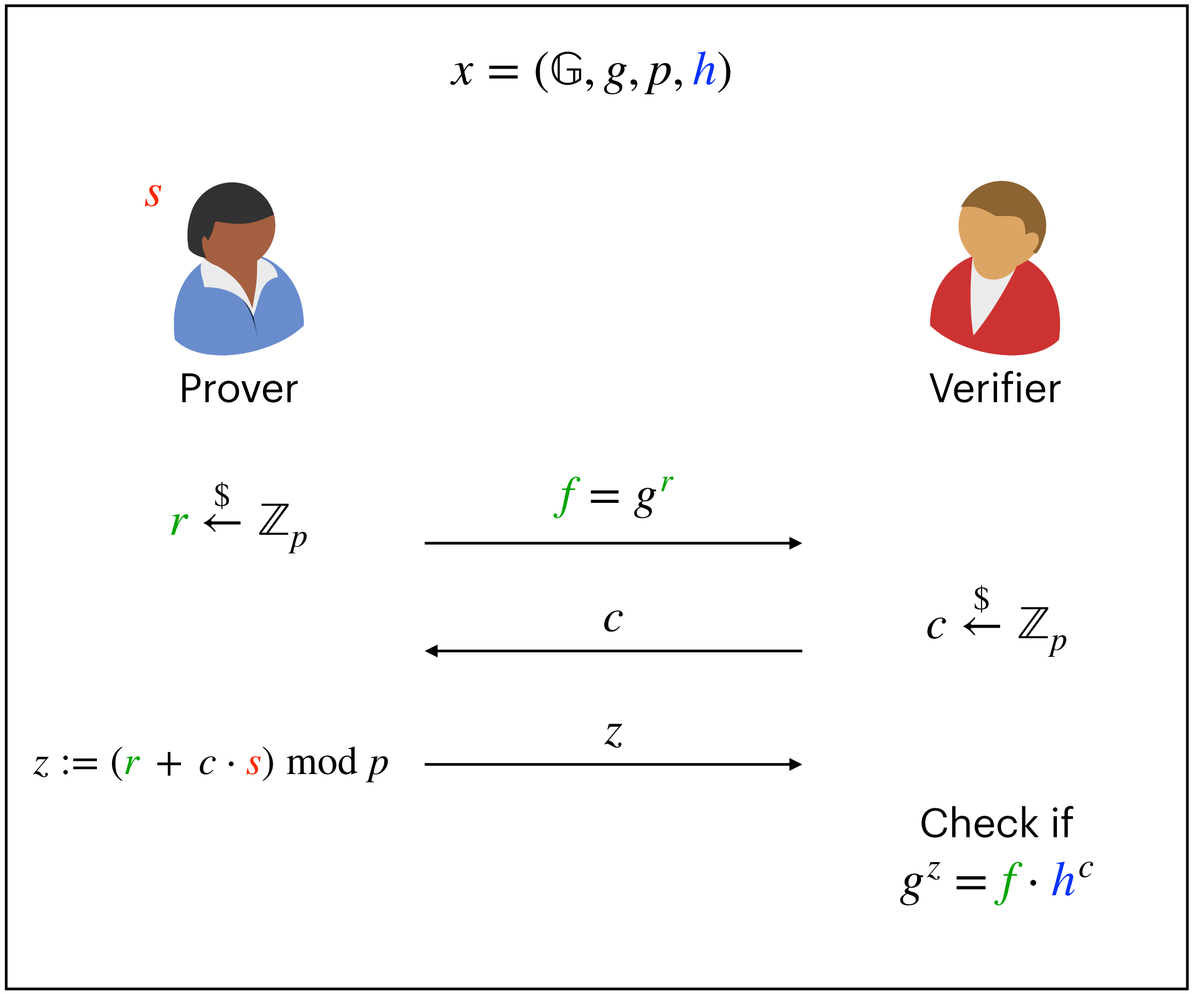

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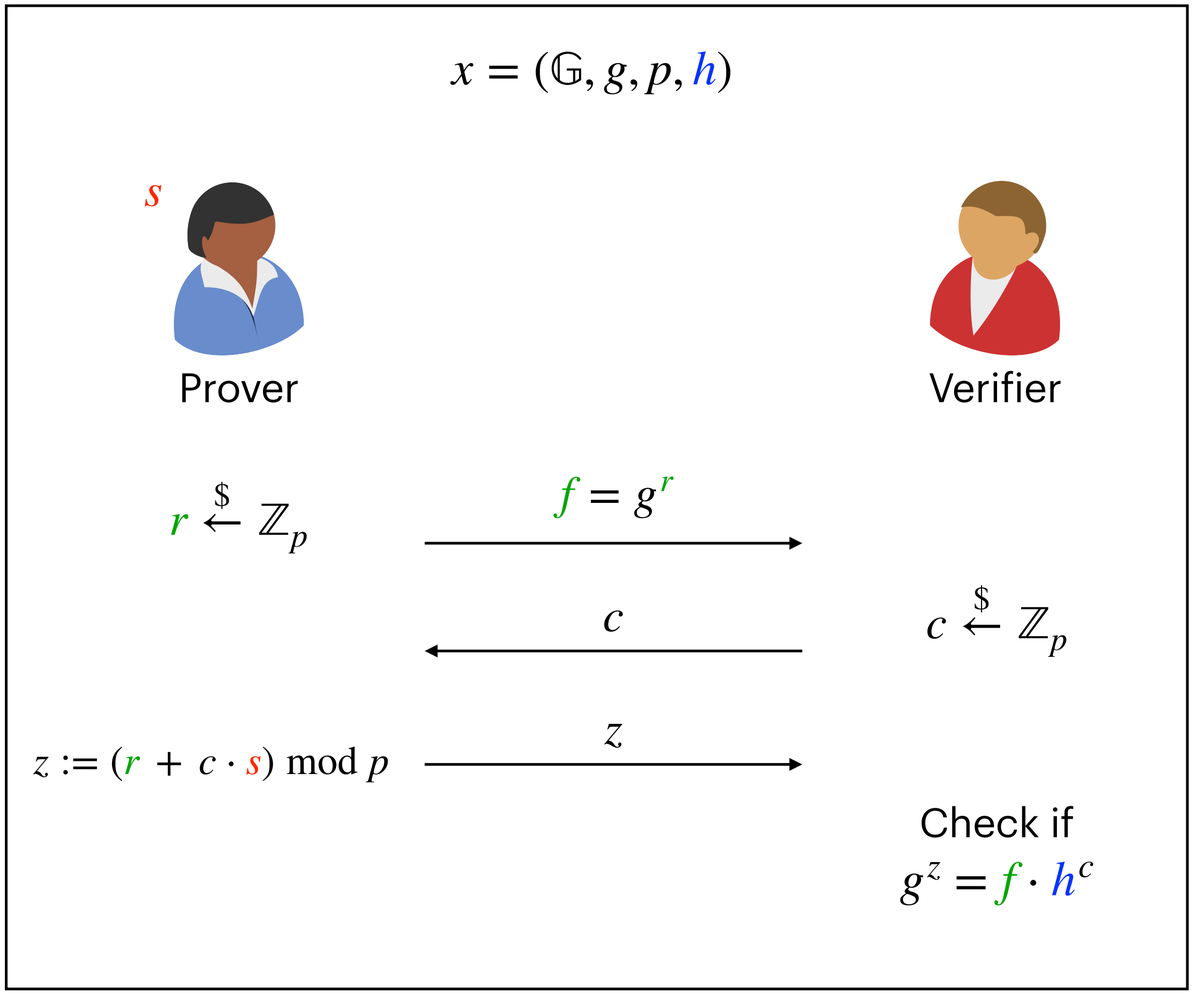
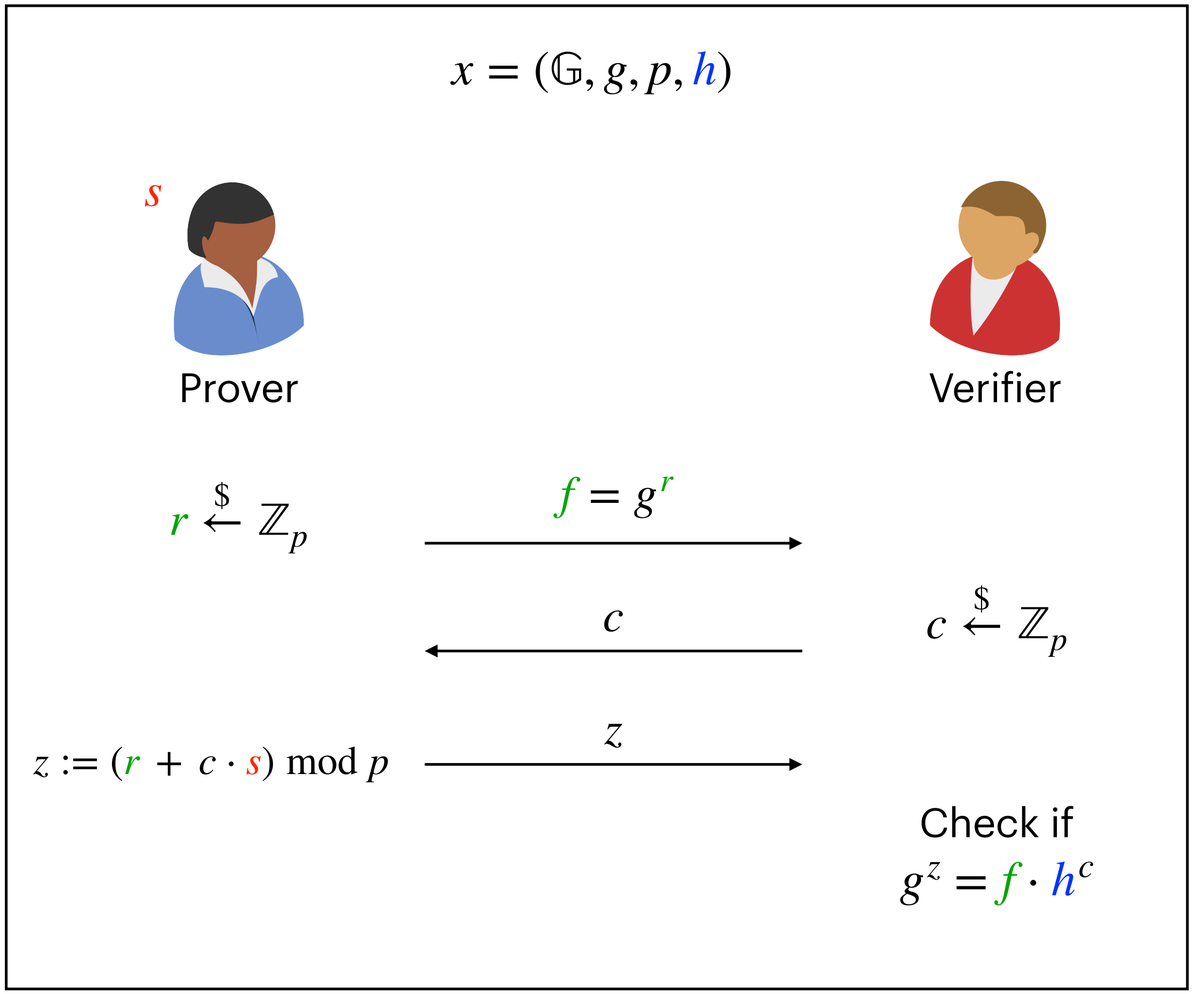


Completeness

$$g^z = g^{r+c\cdot s} = g^r \cdot (g^s)^c = f \cdot h^c .$$

ZK Proof of Knowledge for Discrete Log



Zero-Knowledge

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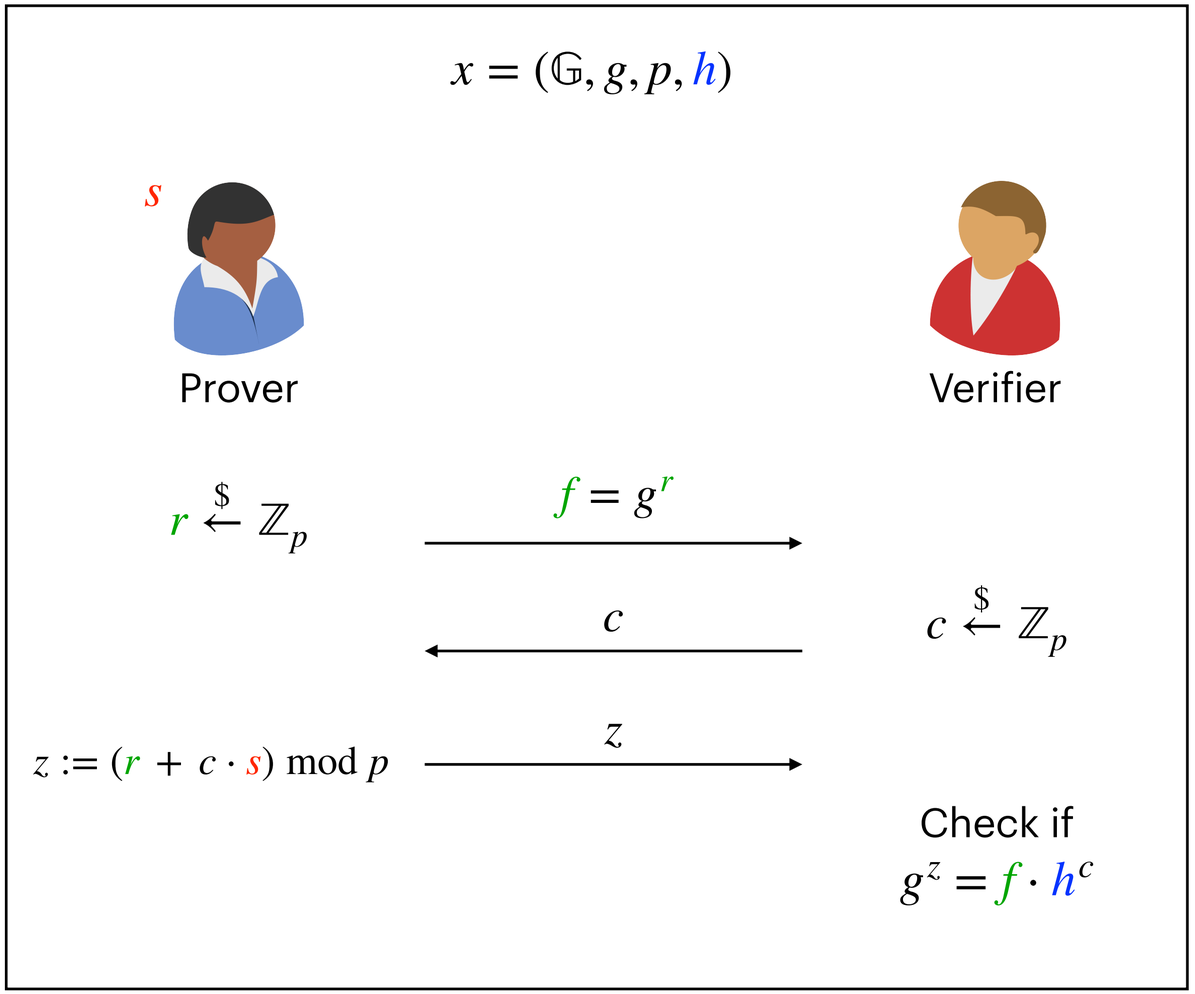

Zero-Knowledge

Not clear how to prove zero-knowledge against arbitrary malicious verifiers.

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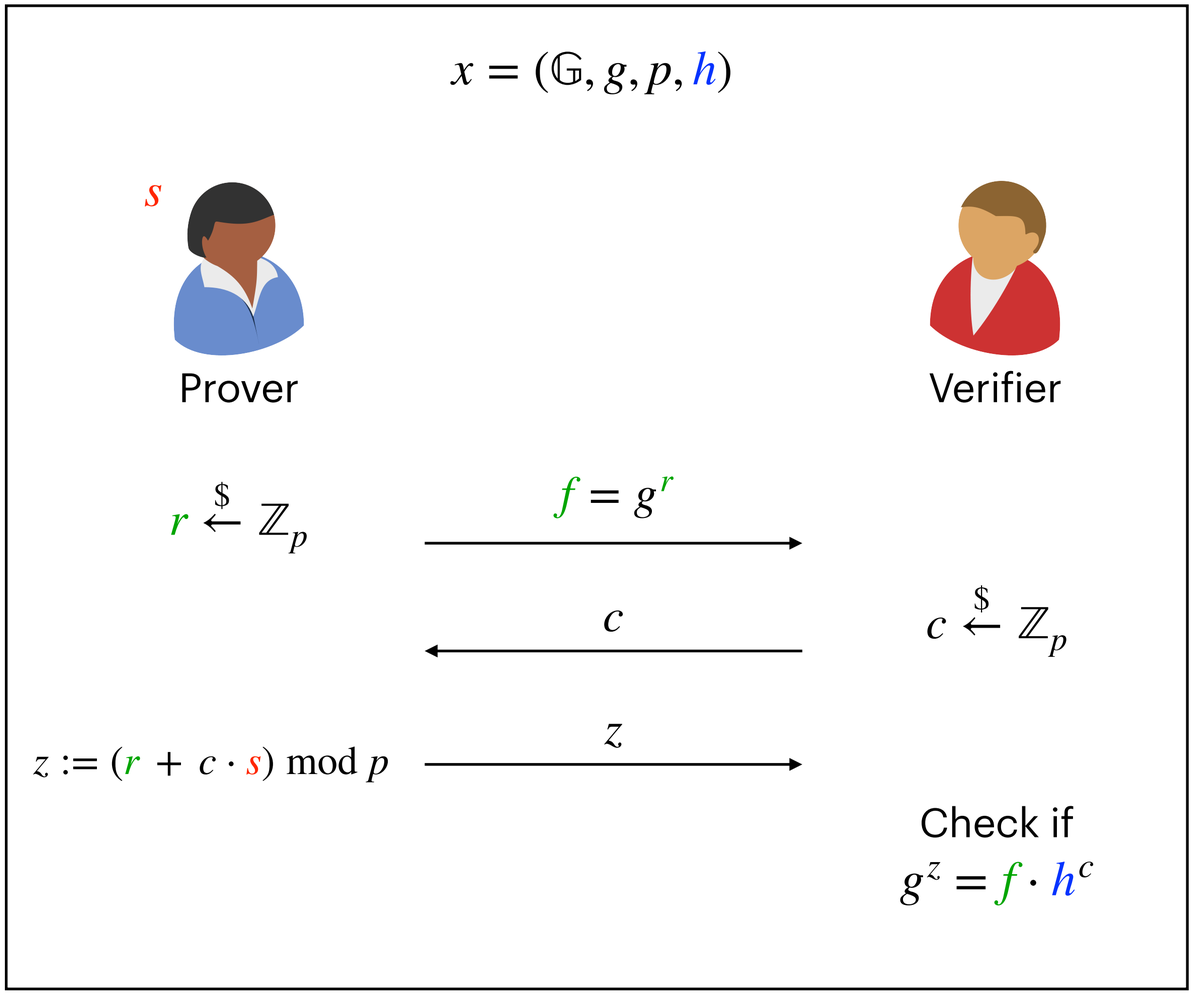

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We will prove a weaker guarantee: an honest verifier will not gain knowledge by interacting with the prover.

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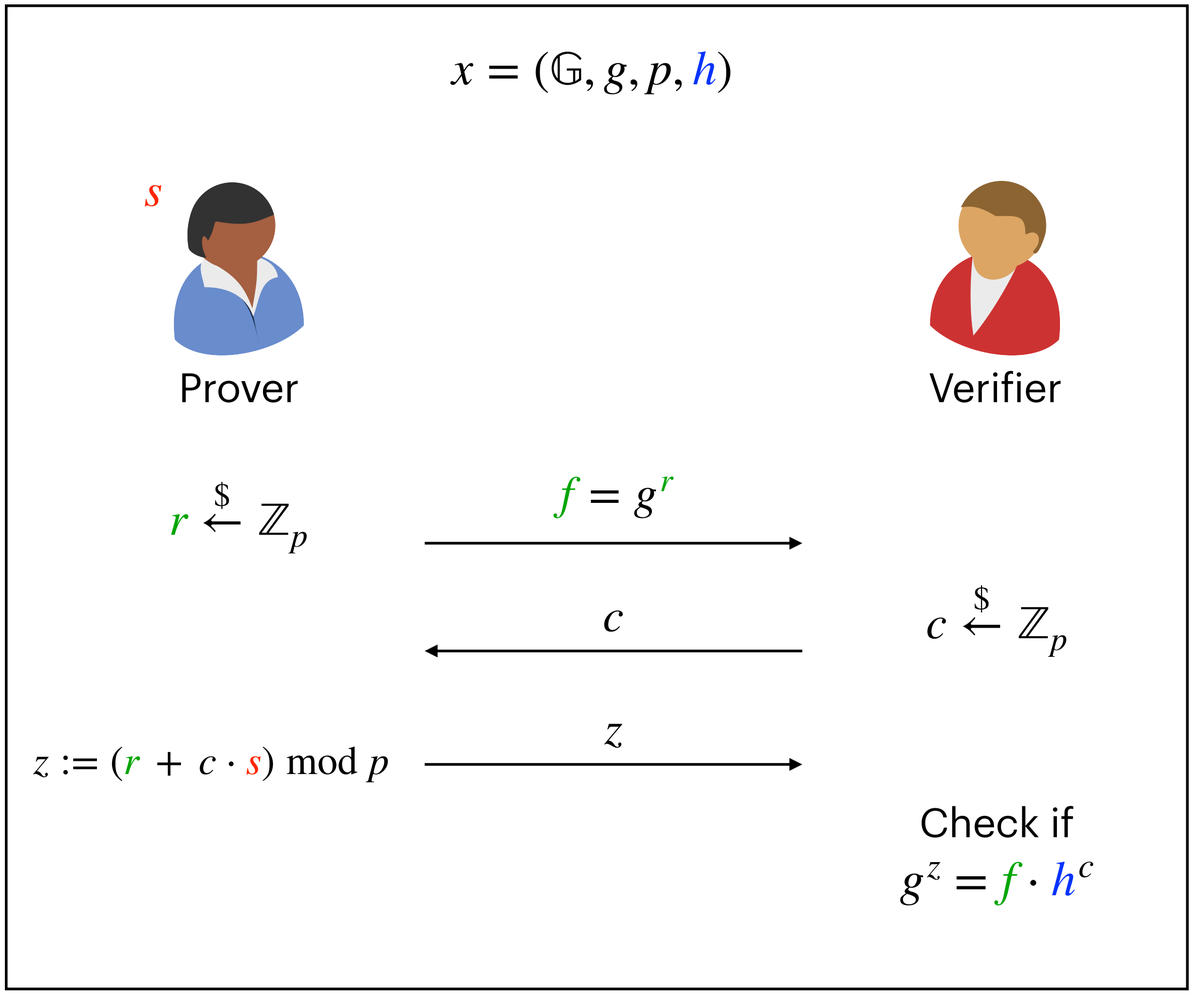

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Honest-Verifier Zero-Knowledge: There is a simulator whose output is indistinguishable from the honest verifier's view in the protocol.

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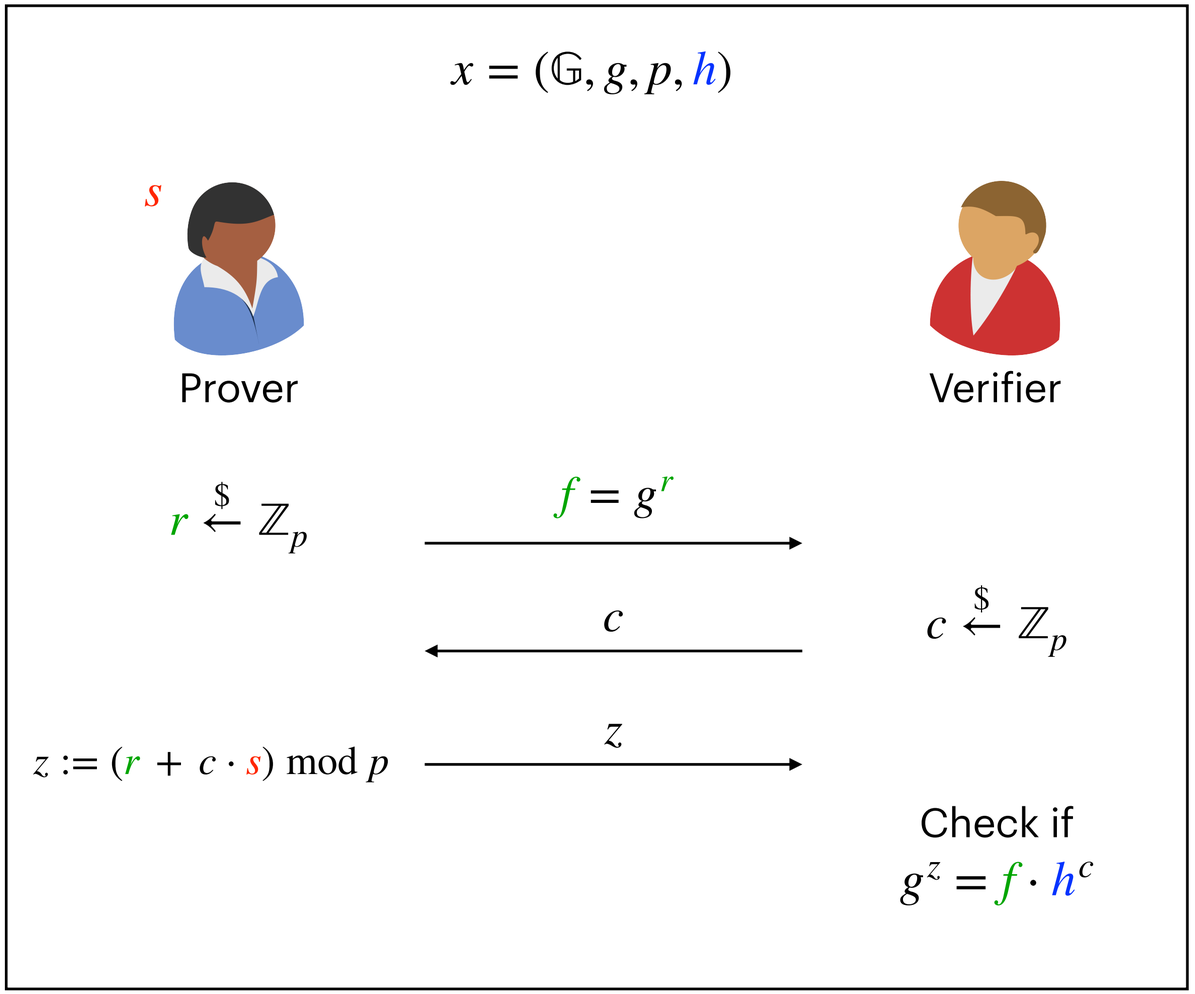

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We will prove a weaker guarantee: an honest verifier will not gain knowledge by interacting with the prover.

Honest-Verifier Zero-Knowledge: There is a simulator whose output is indistinguishable from the honest verifier's view in the protocol.

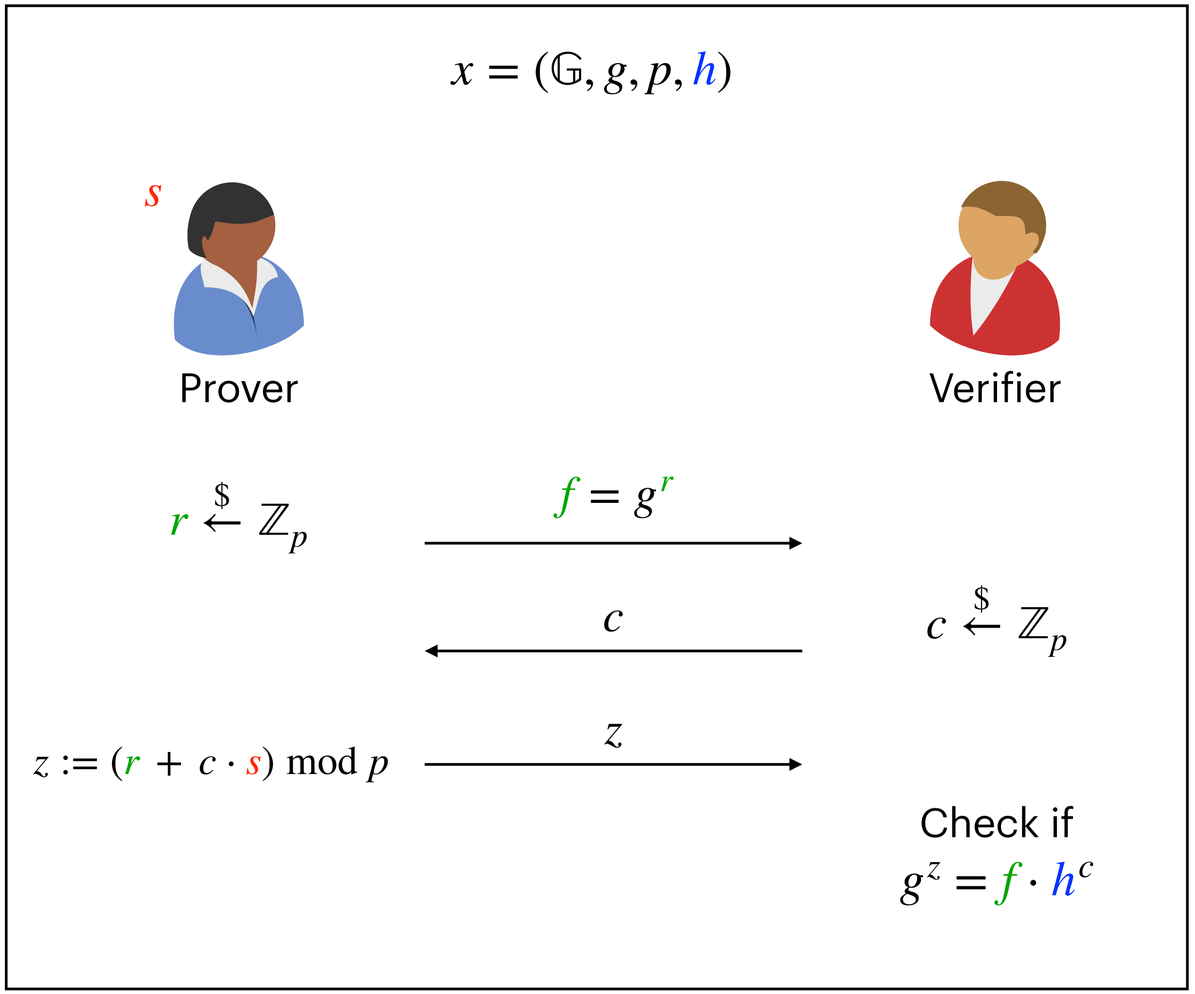
Simulator

$z, c \xleftarrow{\$} \mathbb{Z}_p$

$f = \dfrac{g^z}{h^c}$

Output (f, c, z).

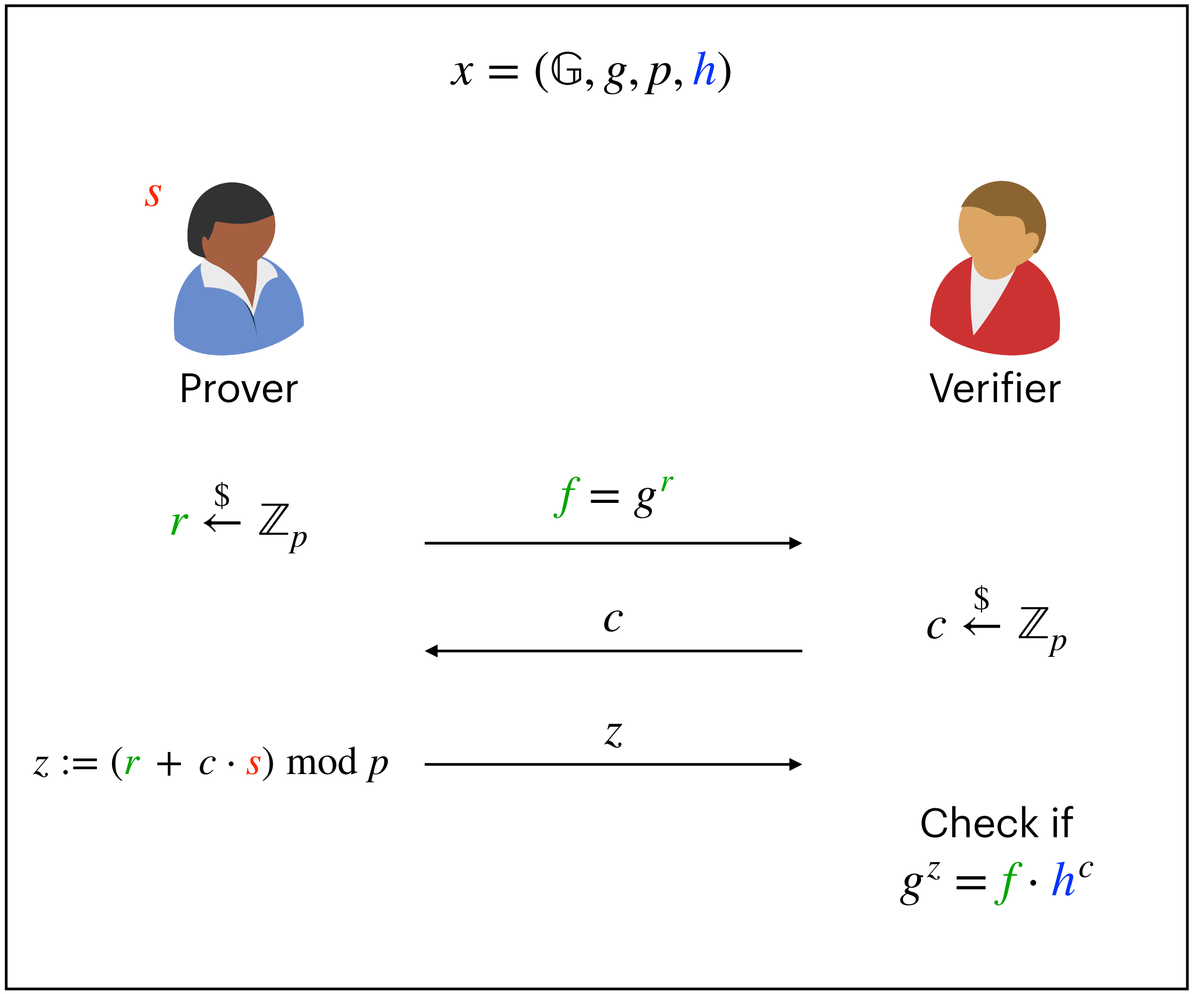
ZK Proof of Knowledge for Discrete Log



Soundness

Prover

ZK Proof of Knowledge for Discrete Log



Soundness


Prover

We will consider the simpler case where the malicious prover convinces the verifier with probability 1.

ZK Proof of Knowledge for Discrete Log

$x = (\mathbb{G}, g, p, h)$

Prover

s

Verifier

$r \xleftarrow{\$} \mathbb{Z}_p$

$f = g^r$

c

$c \xleftarrow{\$} \mathbb{Z}_p$

$z := (r + c \cdot s) \bmod p$

z

Check if

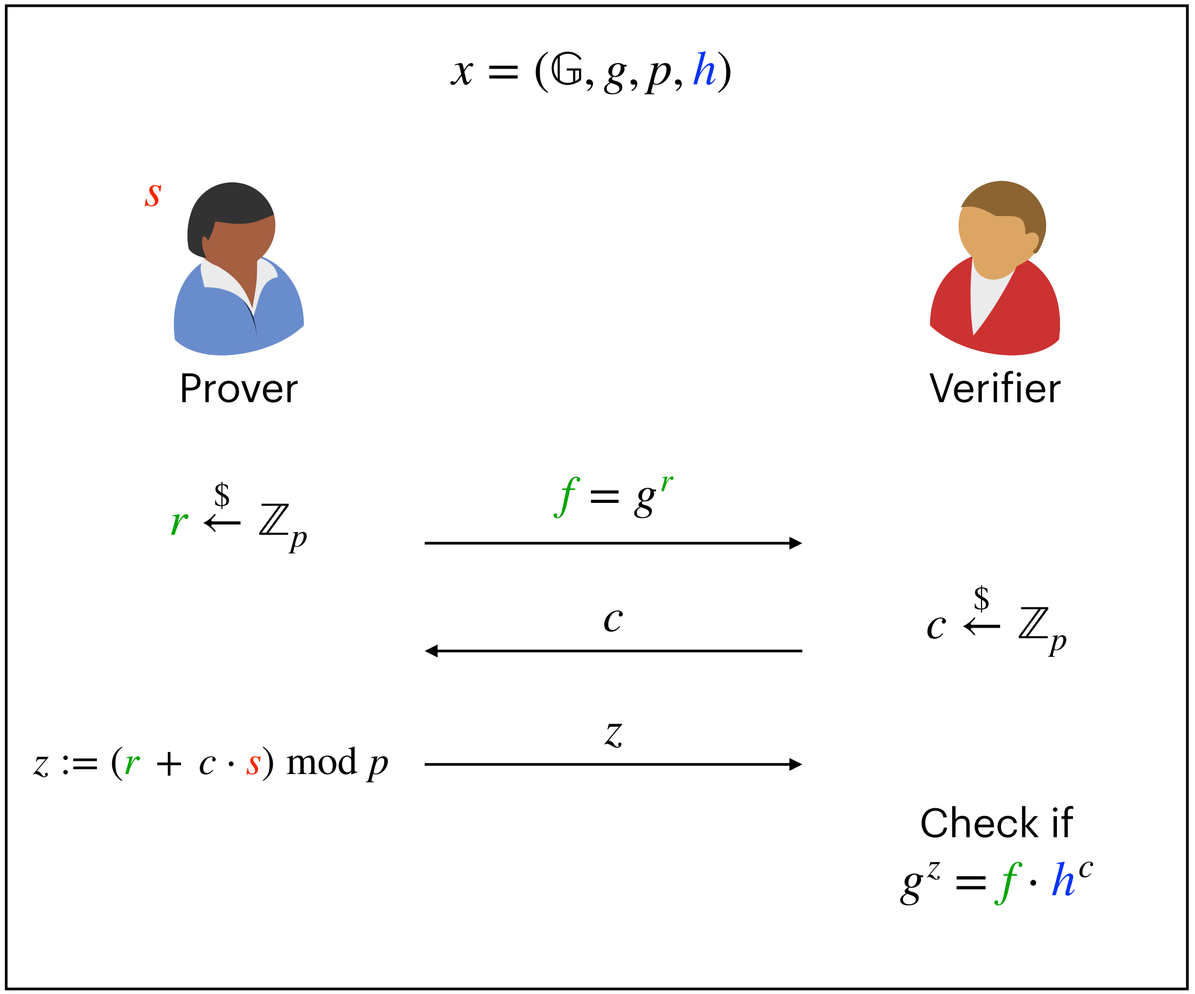
$g^z = f \cdot h^c$

Soundness

Prover

Extractor

ZK Proof of Knowledge for Discrete Log



Soundness


Prover


Extractor

Extract the witness while making the prover believe it is talking to the verifier.

ZK Proof of Knowledge for Discrete Log

$x = (\mathbb{G}, g, p, h)$

Prover

s

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$r \xleftarrow{\$} \mathbb{Z}_p$

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ZK Proof of Knowledge for Discrete Log

$x = (\mathbb{G}, g, p, h)$

Prover

Verifier

s

$r \xleftarrow{\$} \mathbb{Z}_p$

$f = g^r$ →

← c

$c \xleftarrow{\$} \mathbb{Z}_p$

$z := (r + c \cdot s) \bmod p$

z →

Check if $g^z = f \cdot h^c$

Soundness

Prover

f →

Extractor

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z →

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Soundness

Prover

Extractor

f →

← c_1

$c_1 \xleftarrow{\$} \mathbb{Z}_p$

ZK Proof of Knowledge for Discrete Log

$x = (\mathbb{G}, g, p, h)$

s

Prover

Verifier

$r \xleftarrow{\$} \mathbb{Z}_p$

$f = g^r$ →

← c

$c \xleftarrow{\$} \mathbb{Z}_p$

$z := (r + c \cdot s) \bmod p$

z →

Check if $g^z = f \cdot h^c$

Soundness

Prover

Extractor

f →

← c_1

$c_1 \xleftarrow{\$} \mathbb{Z}_p$

z_1 →

$g^{z_1} = f \cdot h^{c_1}$

ZK Proof of Knowledge for Discrete Log

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z →

Check if $g^z = f \cdot h^c$

Soundness

Prover

(f, c_1, z_1)

$g^{z_1} = f \cdot h^{c_1}$

Extractor

f →

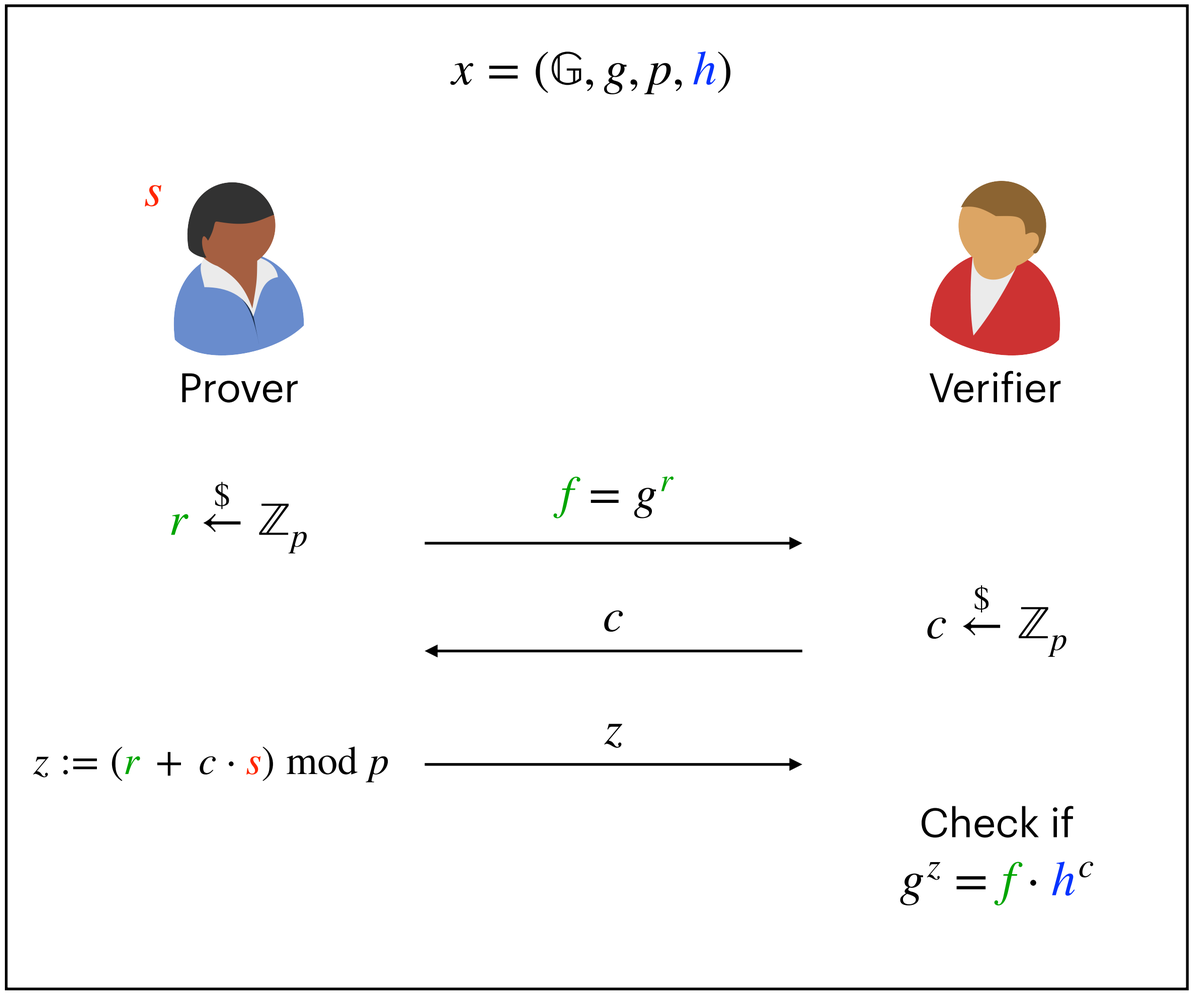
← c_1

$c_1 \xleftarrow{\$} \mathbb{Z}_p$

z_1 →

$g^{z_1} = f \cdot h^{c_1}$

ZK Proof of Knowledge for Discrete Log



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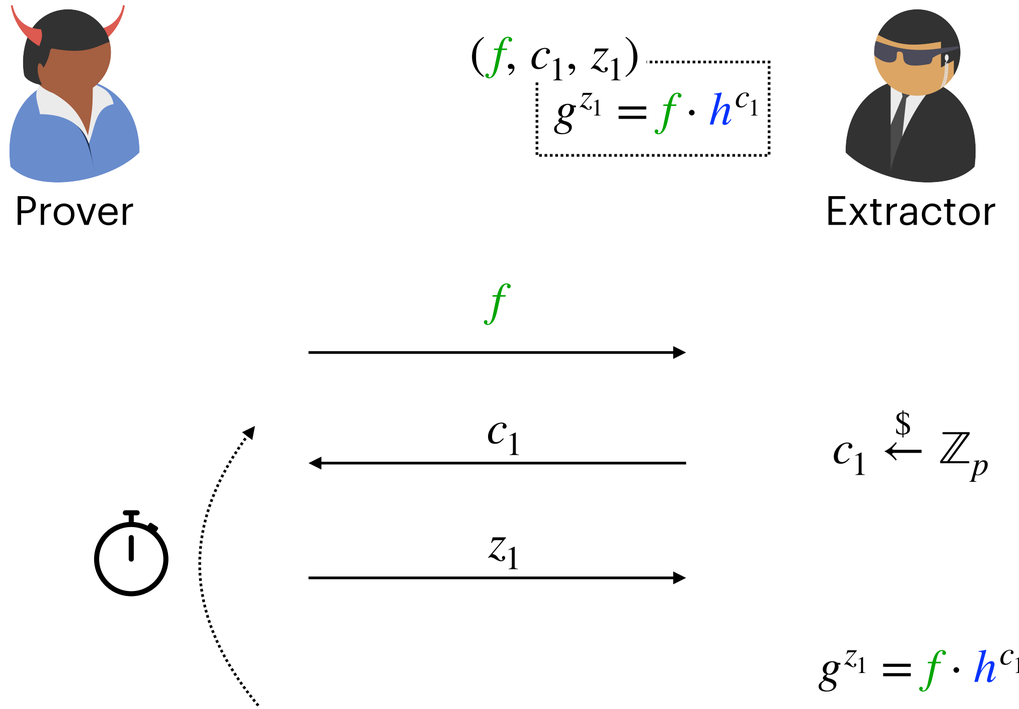

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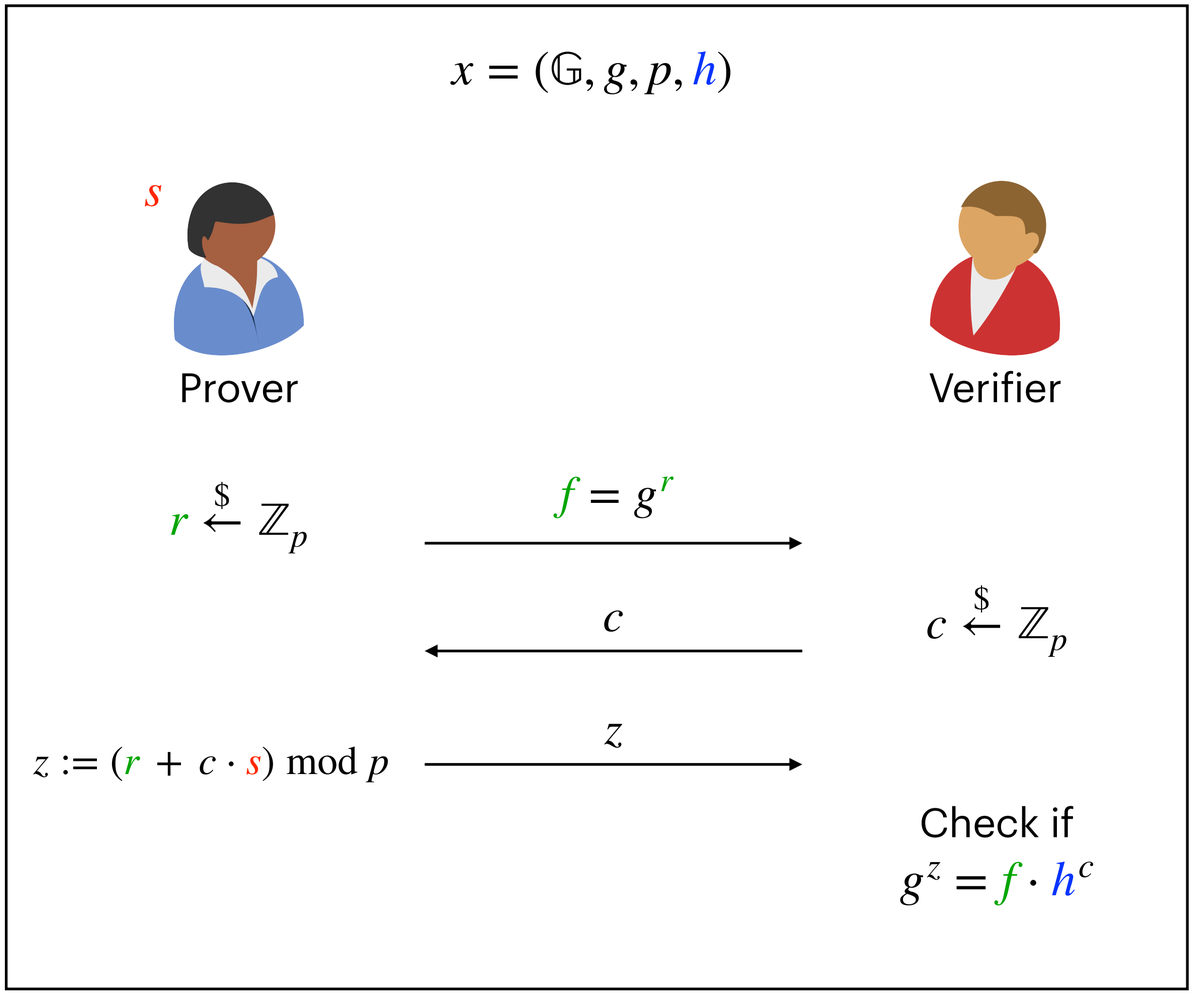

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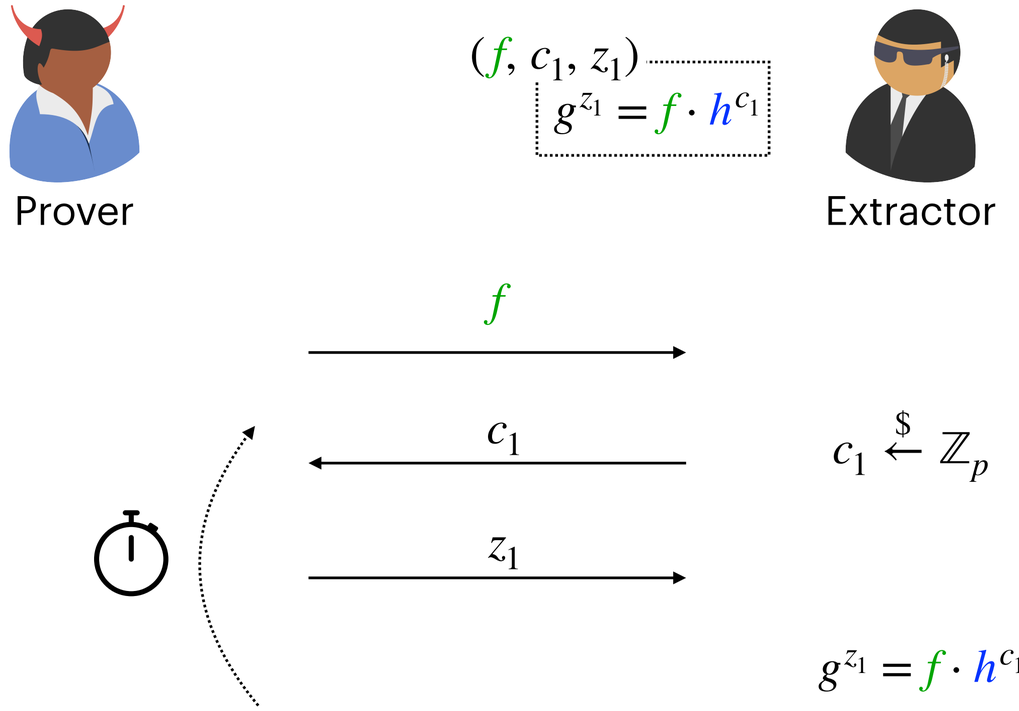

Rewind prover to the first message.

The prover is simply a program. A snapshot of its state can be saved and copied at any point during its execution.

ZK Proof of Knowledge for Discrete Log

$x = (\mathbb{G}, g, p, h)$

s

Prover

Verifier

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$f = g^r$ →

← c

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z →

Check if $g^z = f \cdot h^c$

Soundness

Prover

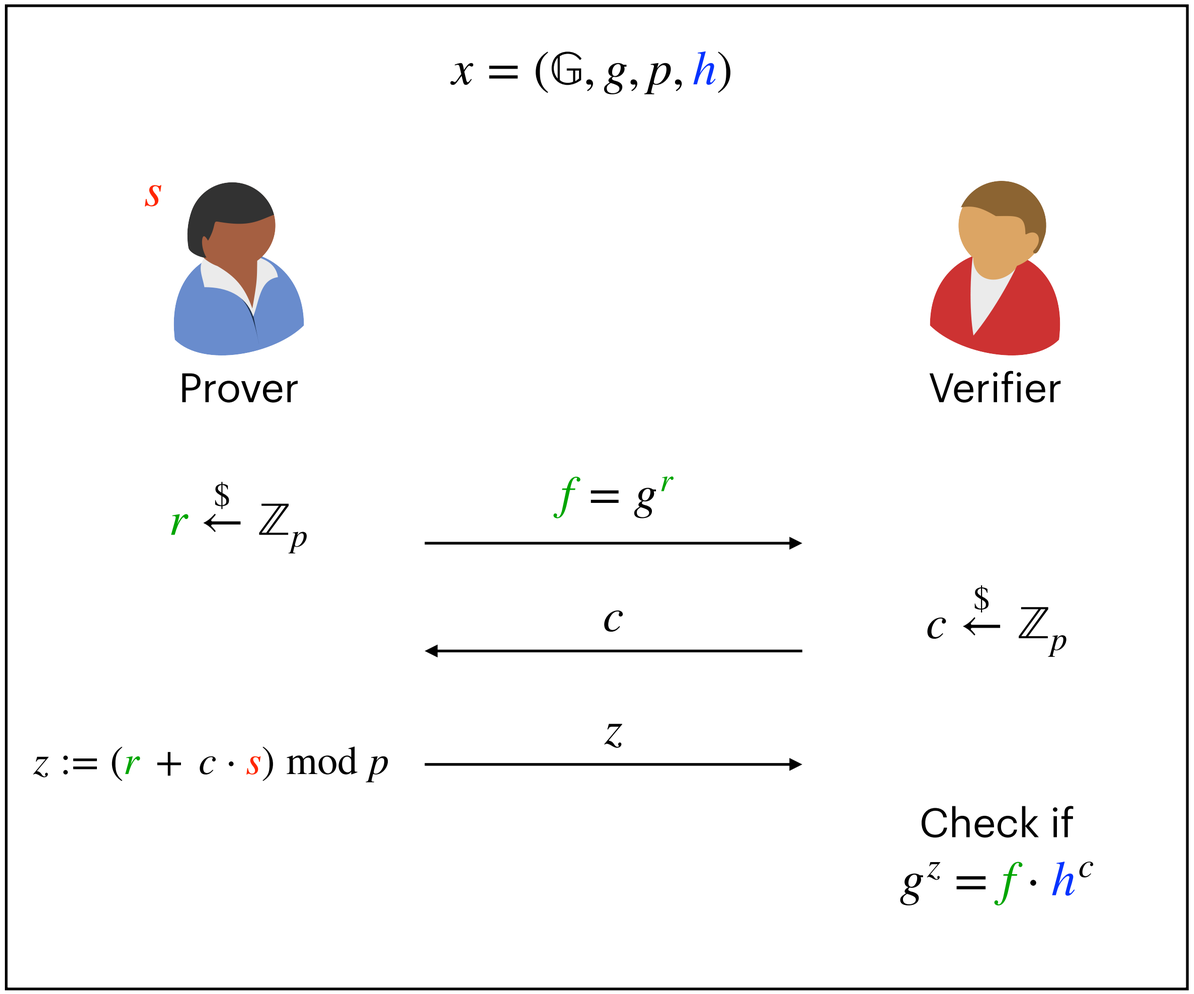
f →

(f, c_1, z_1)

$g^{z_1} = f \cdot h^{c_1}$

Extractor

ZK Proof of Knowledge for Discrete Log



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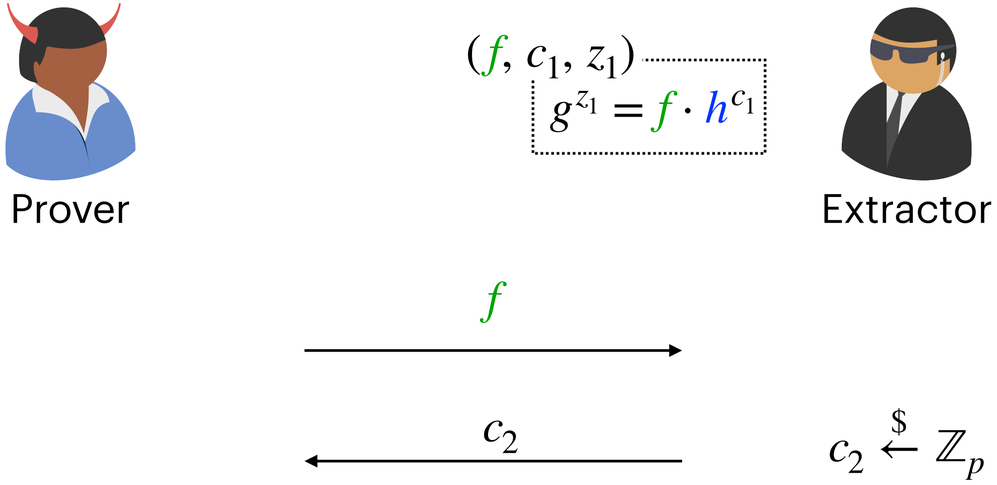

ZK Proof of Knowledge for Discrete Log

$x = (\mathbb{G}, g, p, h)$

s

Prover

Verifier

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← c

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z →

Check if $g^z = f \cdot h^c$

Soundness

Prover

(f, c_1, z_1)

$g^{z_1} = f \cdot h^{c_1}$

Extractor

f →

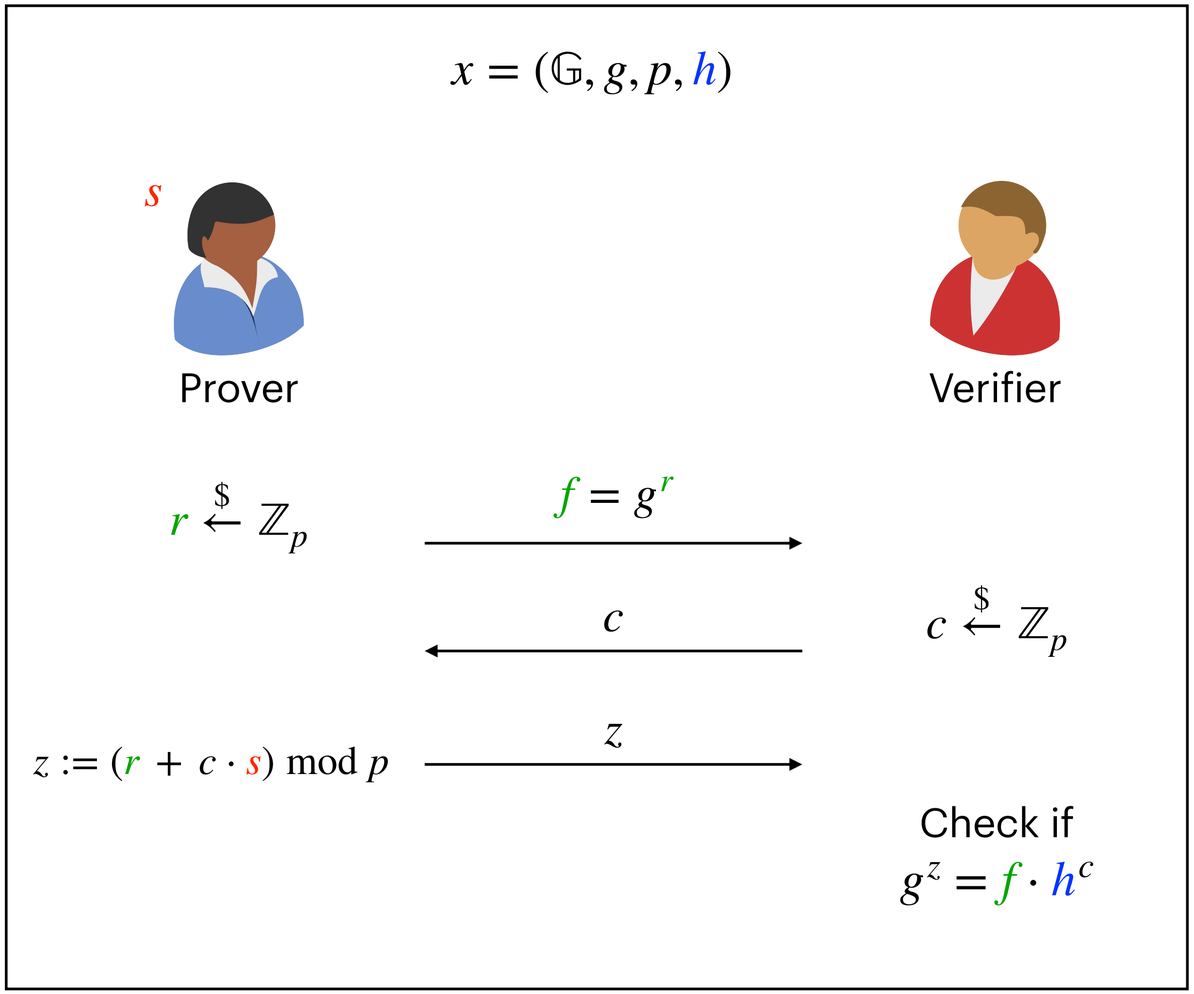
← c_2

$c_2 \xleftarrow{\$} \mathbb{Z}_p$

z_2 →

$g^{z_2} = f \cdot h^{c_2}$

ZK Proof of Knowledge for Discrete Log



Soundness


Prover


Extractor

(f, c_1, z_1)
$g^{z_1} = f \cdot h^{c_1}$
(f, c_2, z_2)
$g^{z_2} = f \cdot h^{c_2}$

f

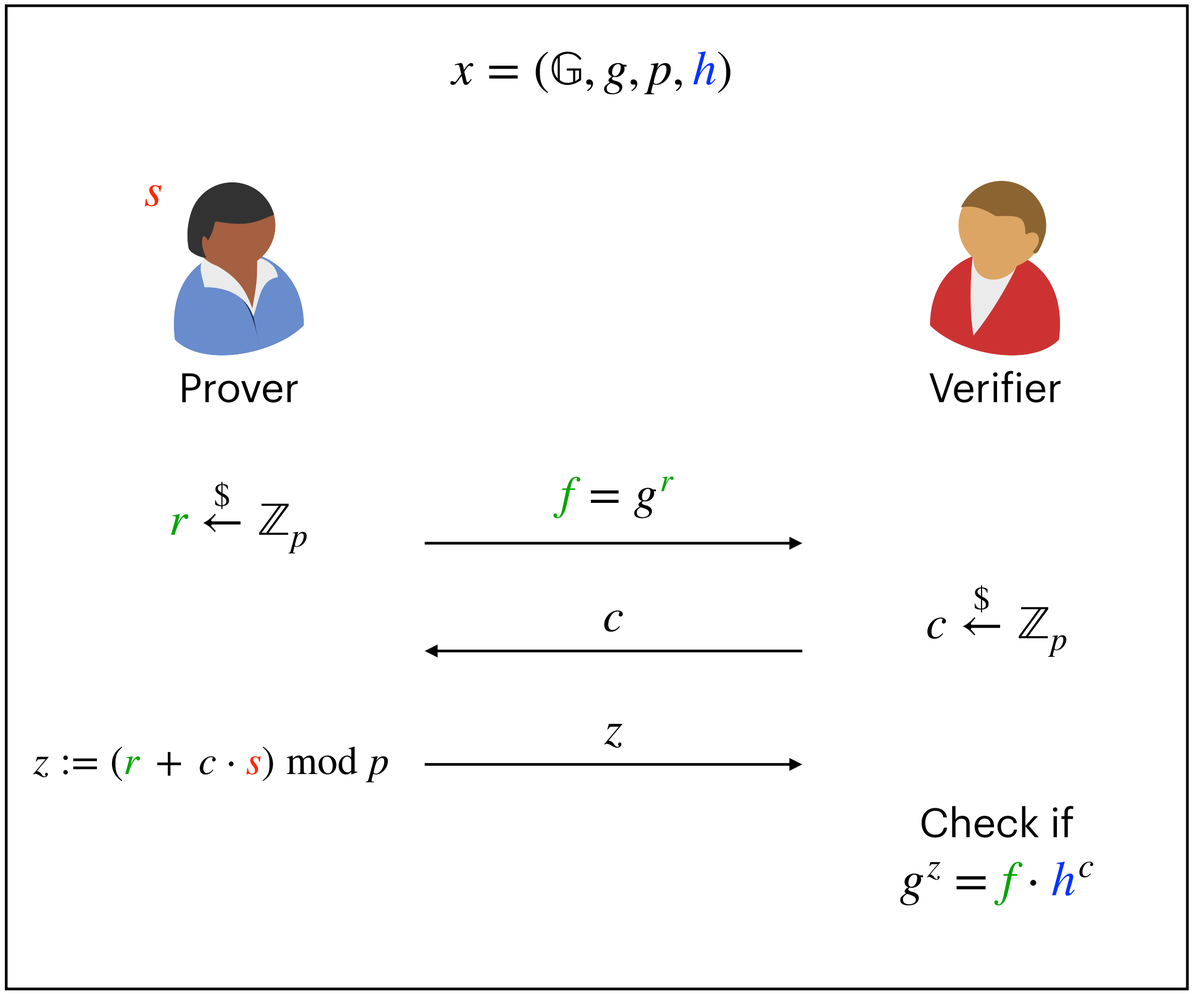
c_2

$c_2 \xleftarrow{\$} \mathbb{Z}_p$

z_2

$g^{z_2} = f \cdot h^{c_2}$

ZK Proof of Knowledge for Discrete Log




Prover

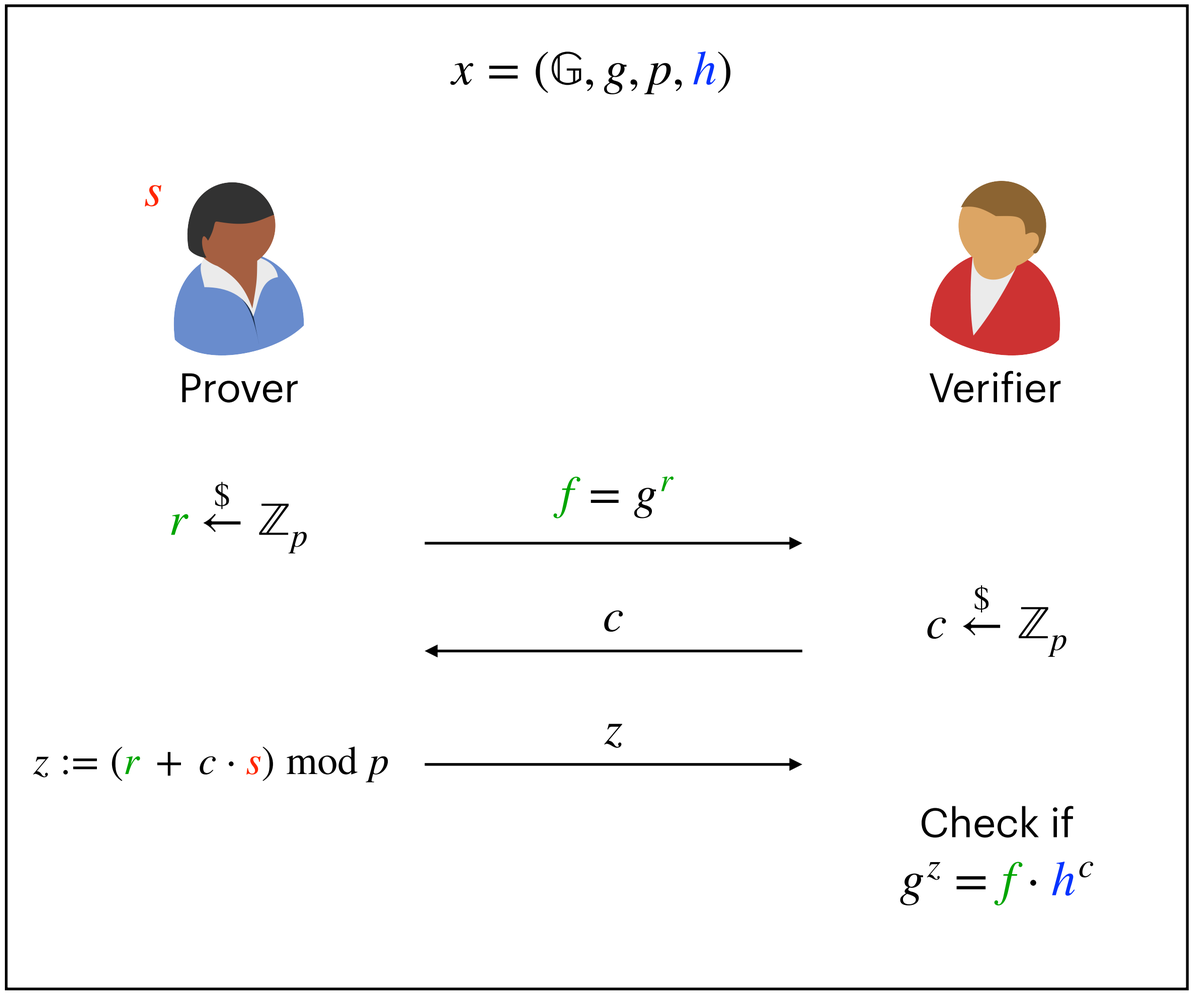
Soundness

(f, c_1, z_1)
$g^{z_1} = f \cdot h^{c_1}$

(f, c_2, z_2)
$g^{z_2} = f \cdot h^{c_2}$


Extractor

ZK Proof of Knowledge for Discrete Log




Prover

Soundness

(f, c_1, z_1)
$g^{z_1} = f \cdot h^{c_1}$

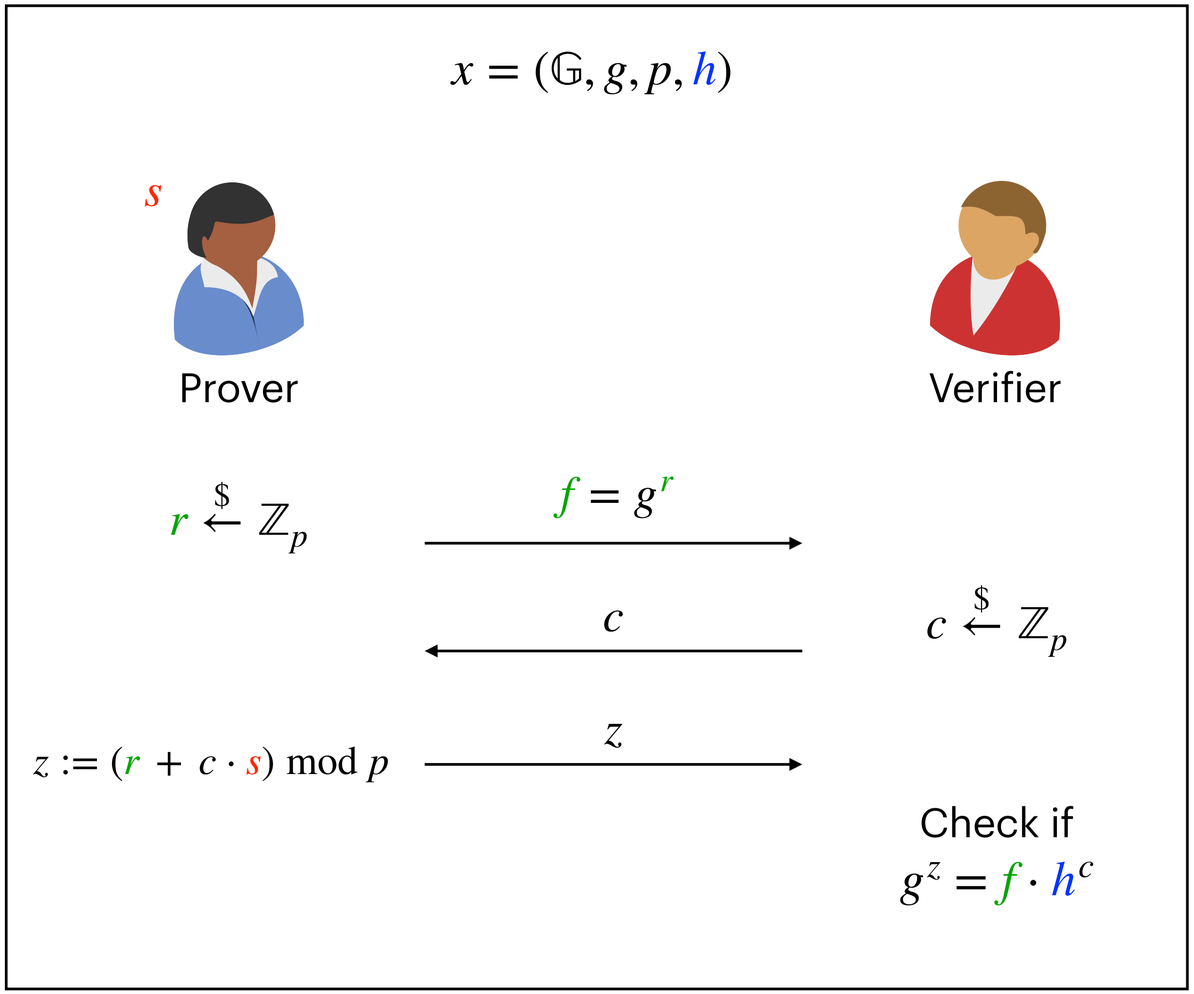
(f, c_2, z_2)
$g^{z_2} = f \cdot h^{c_2}$


Extractor

$\Downarrow$

$g^{z_1 - z_2} = h^{c_1 - c_2}$

ZK Proof of Knowledge for Discrete Log




Prover

Soundness

(f, c_1, z_1) $g^{z_1} = f \cdot h^{c_1}$

(f, c_2, z_2) $g^{z_2} = f \cdot h^{c_2}$


Extractor

$\Downarrow$

$g^{z_1 - z_2} = h^{c_1 - c_2}$

Output $s = \dfrac{z_1 - z_2}{c_1 - c_2} \pmod p$

ZK Proof of Knowledge for Discrete Log

$x = (\mathbb{G}, g, p, h)$

Prover

Verifier

s

$r \xleftarrow{\$} \mathbb{Z}_p$

$f = g^r$ →

← c

$c \xleftarrow{\$} \mathbb{Z}_p$

$z := (r + c \cdot s) \bmod p$

z →

Check if $g^z = f \cdot h^c$

Prover

Soundness

(f, c_1, z_1) $g^{z_1} = f \cdot h^{c_1}$

(f, c_2, z_2) $g^{z_2} = f \cdot h^{c_2}$

Extractor

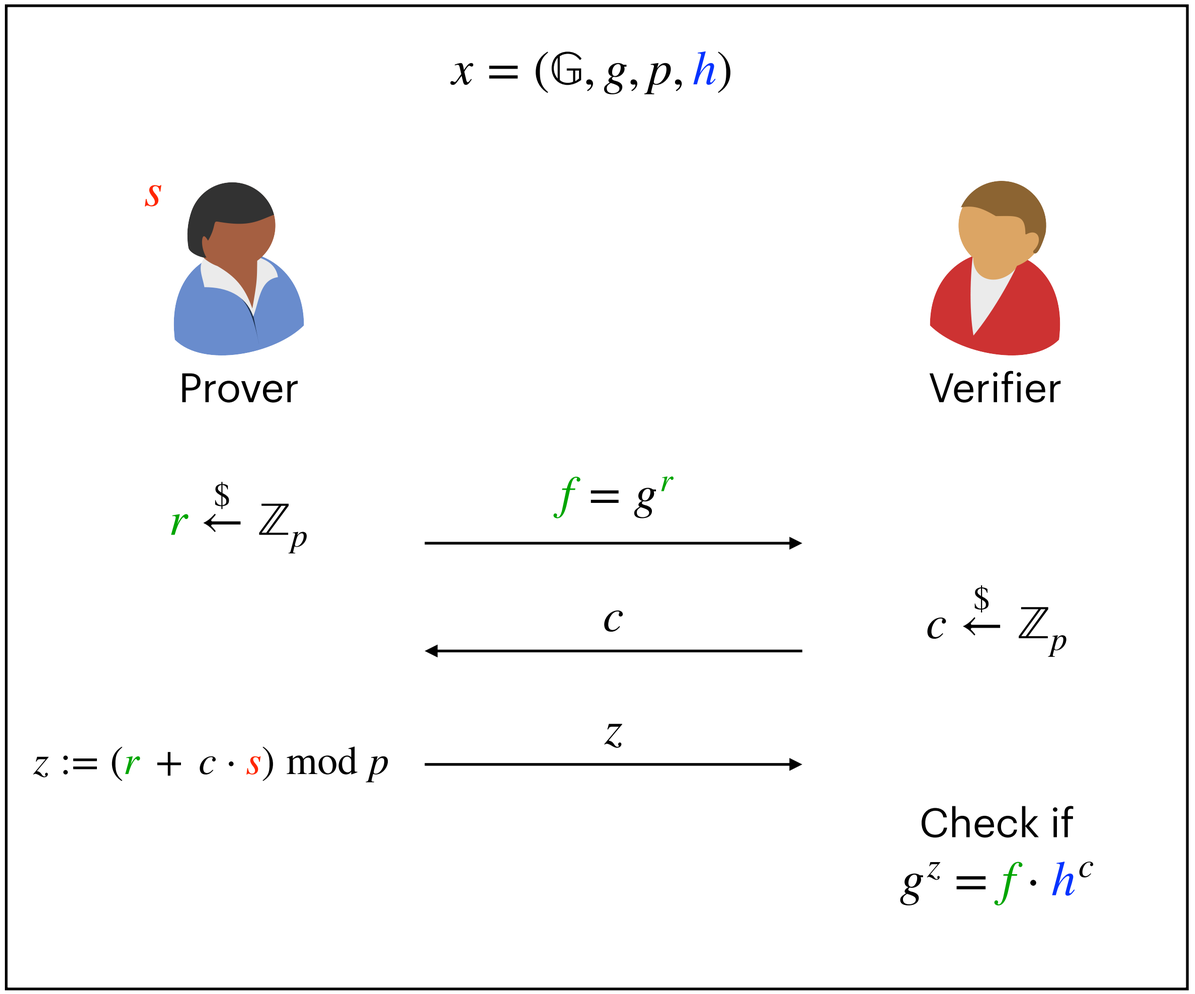
$\Downarrow$

$g^{z_1 - z_2} = h^{c_1 - c_2}$

Output $s = \dfrac{z_1 - z_2}{c_1 - c_2} \pmod p$

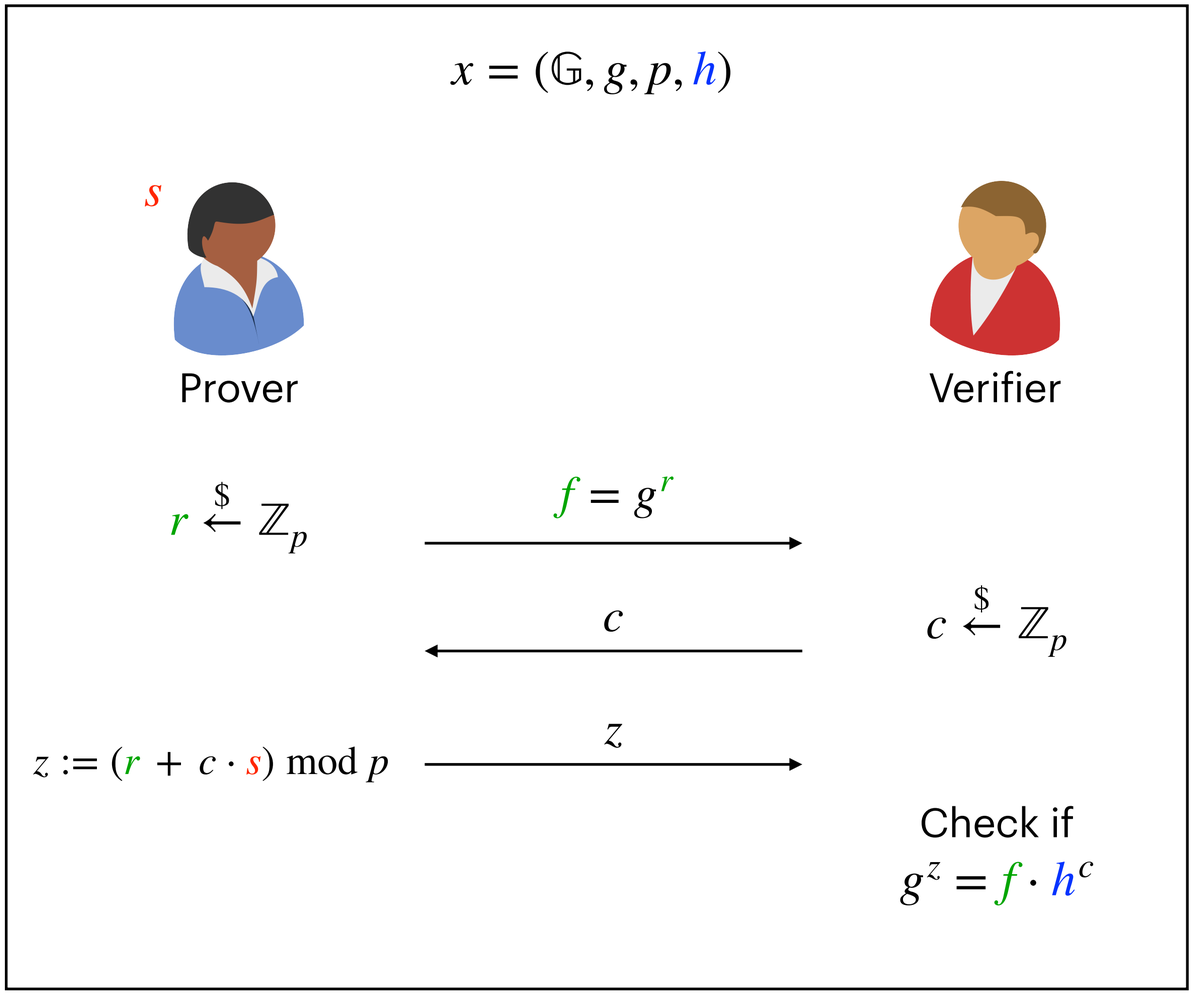
Since $p \in O(2^\lambda)$ probability that $c_1 = c_2$ is negligible.

NIZK Proof of Knowledge for Discrete Log



This proof is public-coin!

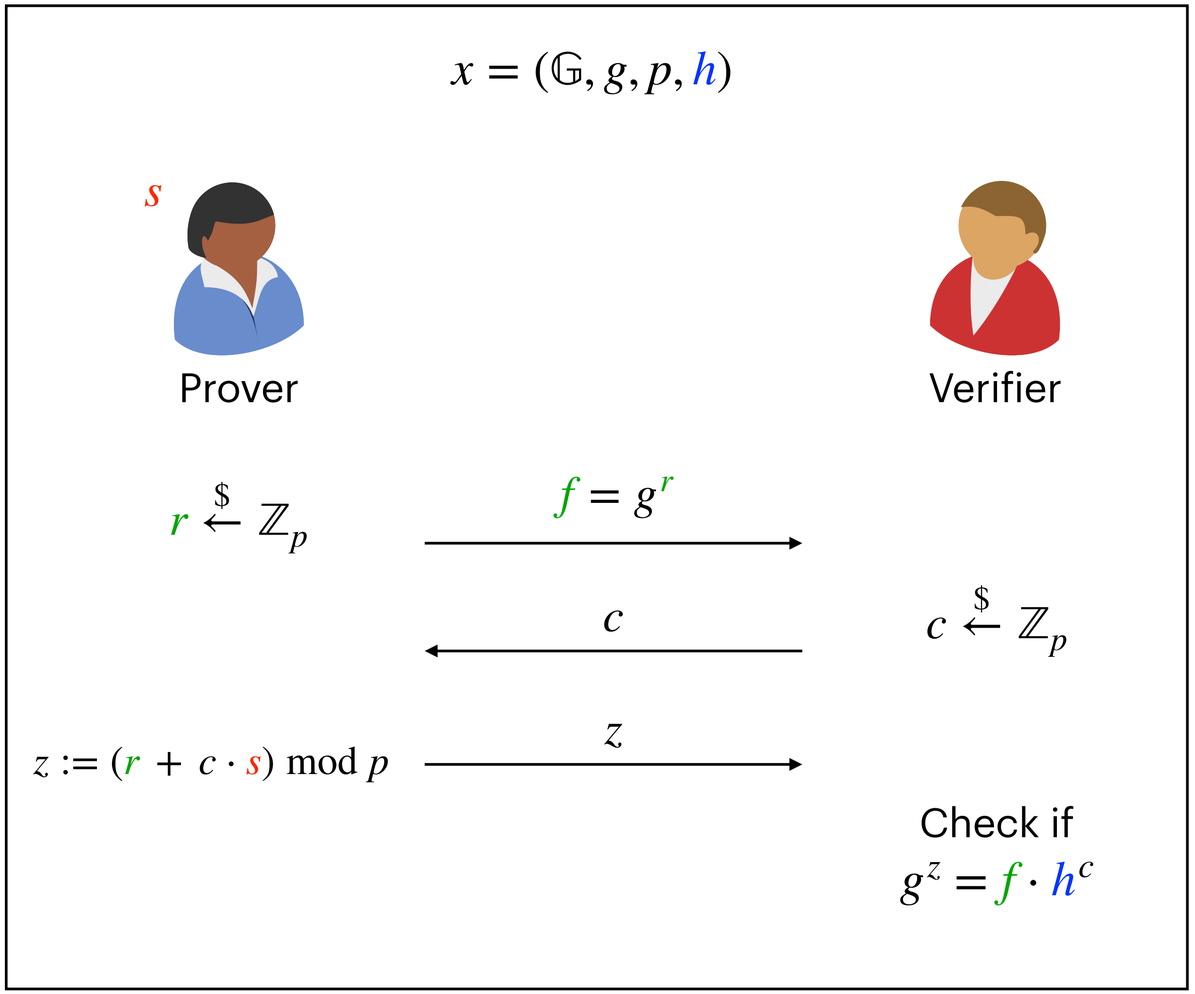
NIZK Proof of Knowledge for Discrete Log



This proof is public-coin!

We can use Fiat-Shamir to transform it into a NIZK in the random oracle model.

NIZK Proof of Knowledge for Discrete Log



This proof is public-coin!

We can use Fiat-Shamir to transform it into a NIZK in the random oracle model.

The NIZK will in fact be a proof of knowledge!

NIZK Proof of Knowledge for Discrete Log

$x = (\mathbb{G}, g, p, h)$

s

Prover

Verifier

$r \xleftarrow{\$} \mathbb{Z}_p$

$f = g^r$

$c := H(x, f)$

$z := (r + c \cdot s) \bmod p$

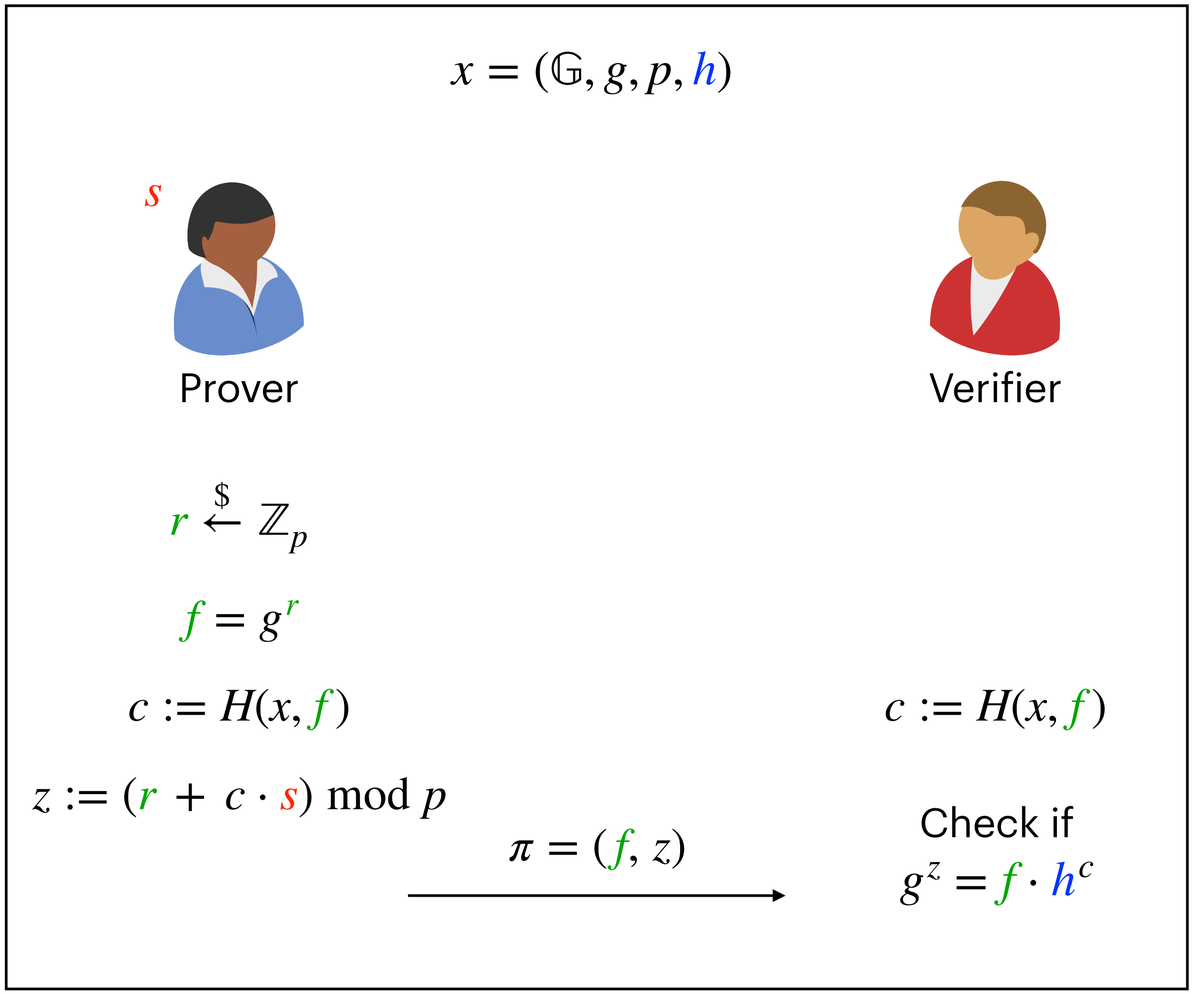
$\pi = (f, z)$

$c := H(x, f)$

Check if

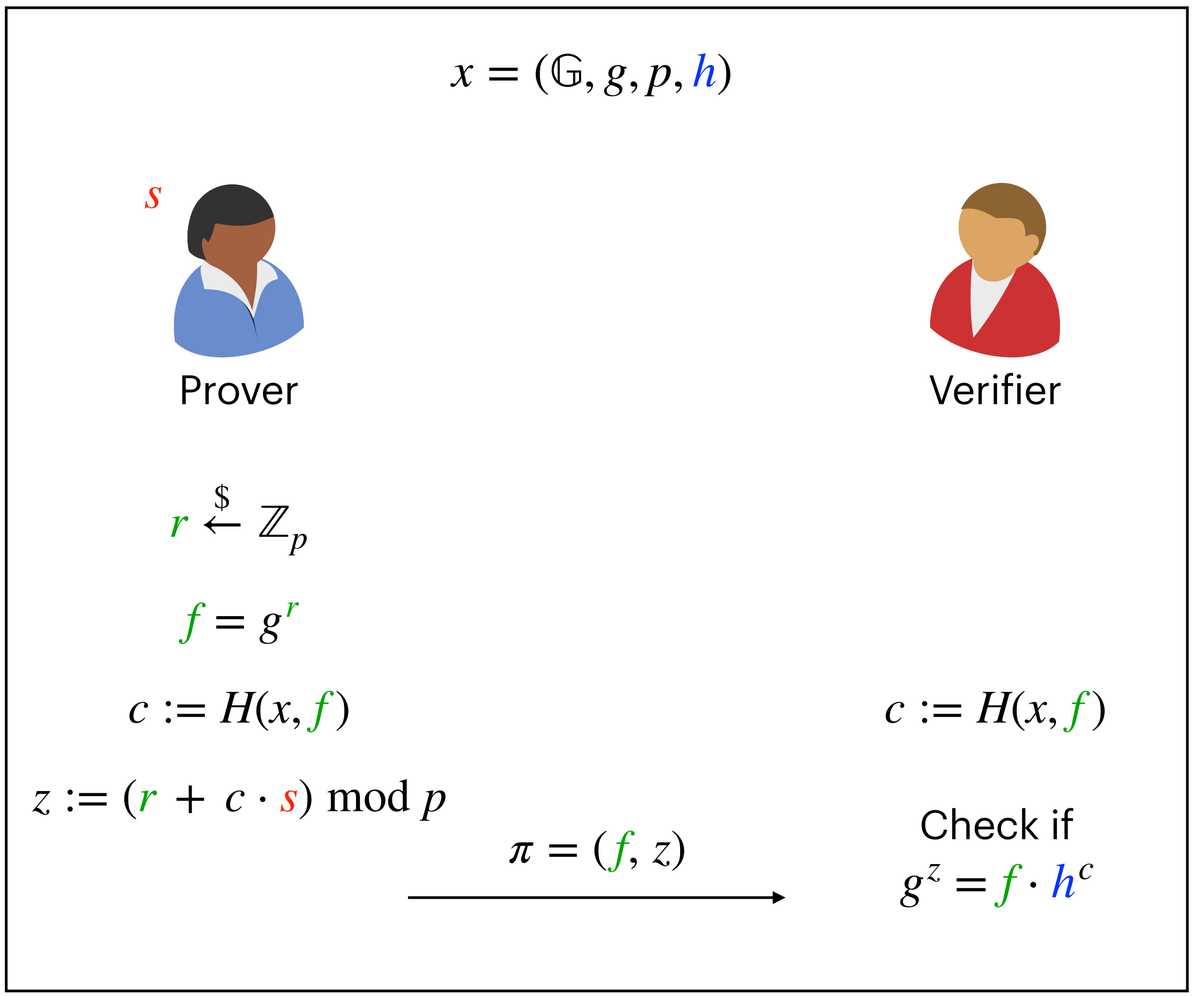
$g^z = f \cdot h^c$

NIZK Proof of Knowledge for Discrete Log



This proves that Alice knows the secret key s for the public key h.

NIZK Proof of Knowledge for Discrete Log



This proves that Alice knows the secret key s for the public key h.

Can be converted into a signature scheme by additionally including the message to be signed in the hash.

Schnorr Signature Scheme

$x = (\mathbb{G}, g, p, h)$

s

Prover

Verifier

$r \xleftarrow{\$} \mathbb{Z}_p$

$f = g^r$

$c := H(x, m, f)$ $\qquad$ $c := H(x, m, f)$

$z := (r + c \cdot s) \bmod p$

$\pi = (f, z)$ $\longrightarrow$

Check if $g^z = f \cdot h^c$

This proves that Alice knows the secret key s for the public key h.

Can be converted into a signature scheme by additionally including the message to be signed in the hash.