Limitations of Perfect Security

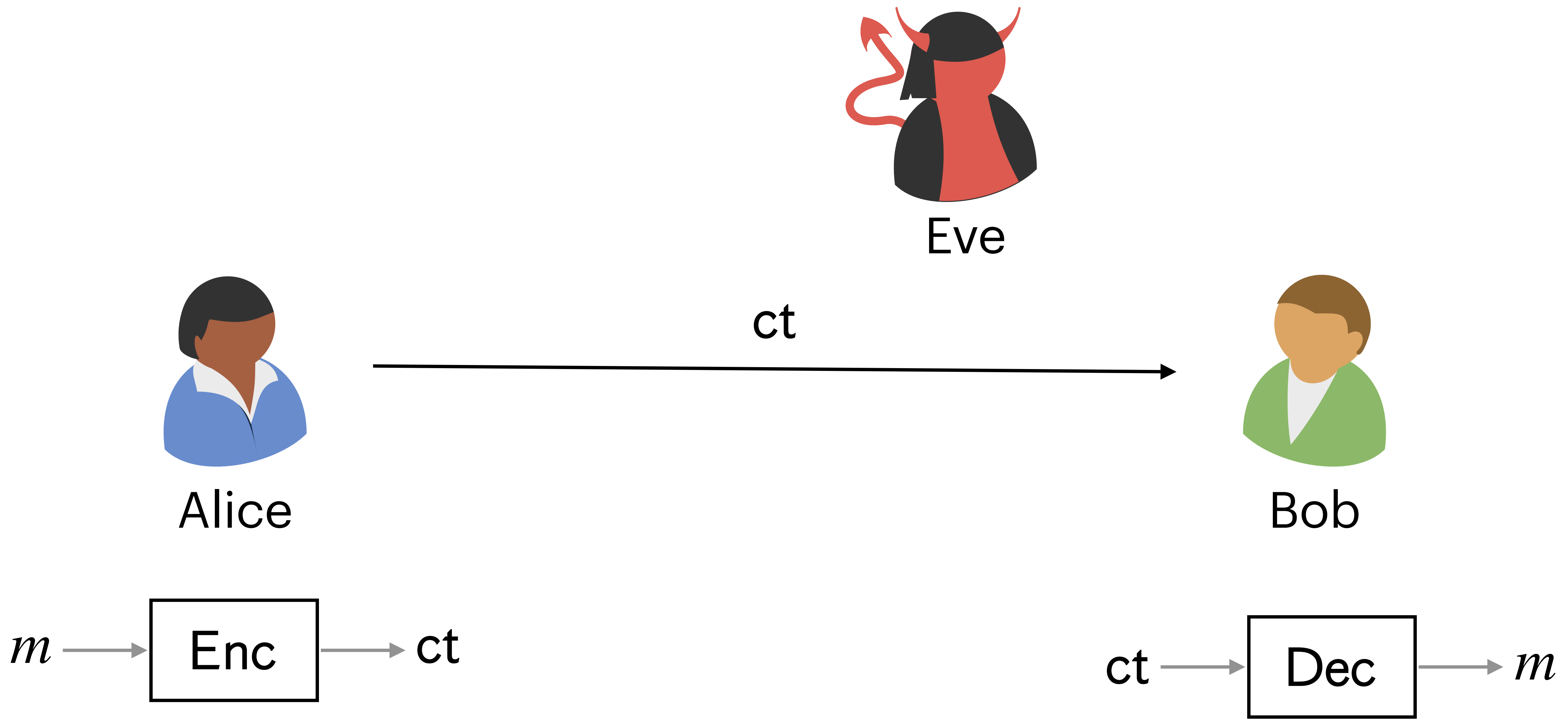
601.442/642 Modern Cryptography

27th January 2026

Announcement

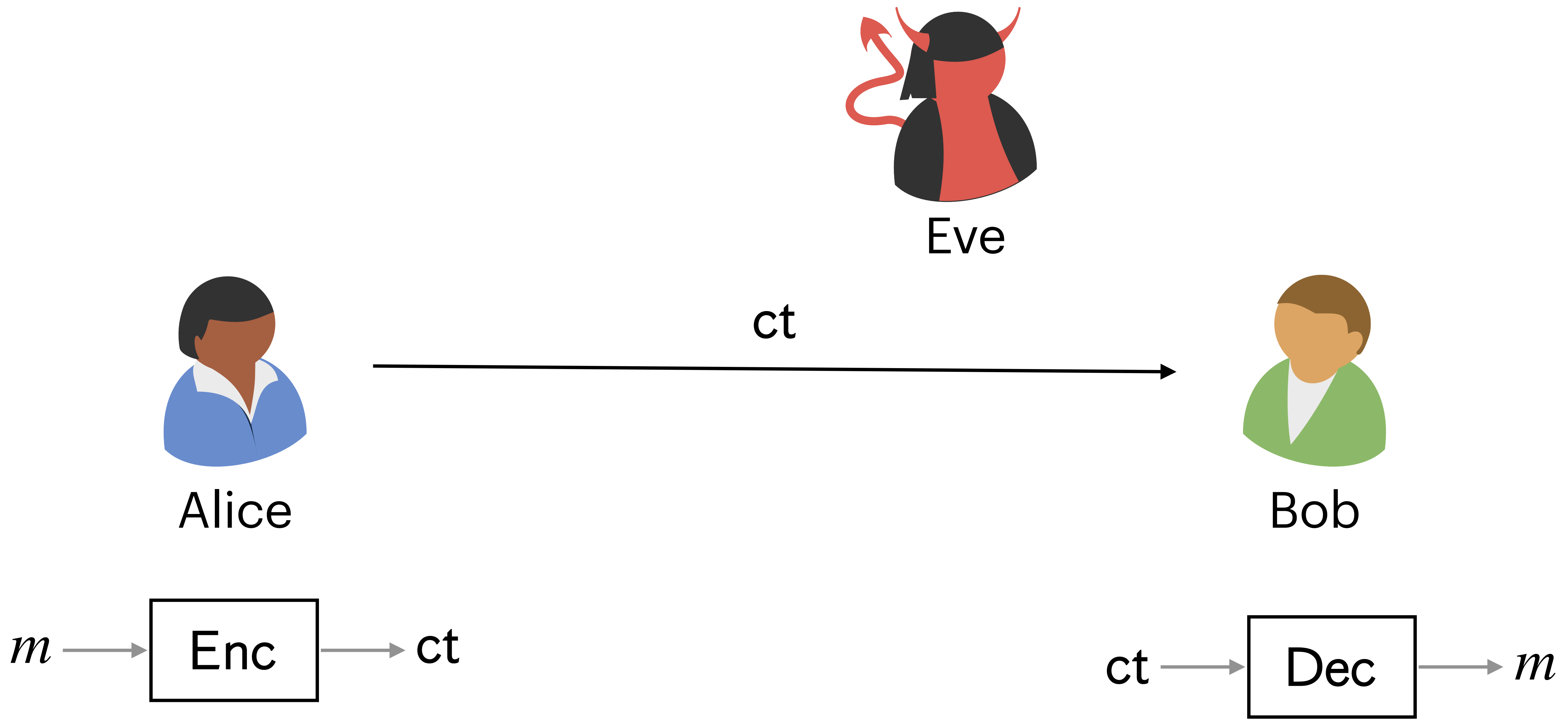
- Homework 1 due this **Thursday** (29th January)
- Please start early and come to office hours with any questions!

Recap: Encryption Scheme



Goal: Bob can decrypt ct but not Eve.

Recap: Encryption Scheme



Goal: Bob can decrypt **ct** but not Eve.

Kerckhoffs' Principle: The security of a cryptosystem shouldn't rely on the secrecy of the algorithm (only the key)

Recap: Encryption Scheme Syntax and Correctness

Encryption Scheme Syntax

An encryption scheme consists of three (possibly probabilistic) algorithms:

- $\mathsf{KeyGen}() \to k$ outputs a key $k \in \mathscr{K}$.
- $\mathsf{Enc}(k, m) \to \mathsf{ct}$ takes key k and message $m \in \mathscr{M}$ and outputs ciphertext $\mathsf{ct} \in \mathscr{C}$.
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Encryption Scheme Correctness

An encryption scheme satisfies correctness if $\forall k \in \mathcal{K}, \forall m \in \mathcal{M}$, we have

$$\Pr[\mathsf{Dec}(k, \mathsf{Enc}(k, m)) = m] = 1,$$

where the probability is over the randomness used in encryption and decryption.

Recap: One-Time Uniform Ciphertext Security

- What the security definition should capture for encryption schemes like OTP
    - The secret key should be kept hidden from Eve.
    - The key is only used to encrypt one plaintext.
    - The ciphertext looks uniformly random to Eve.

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One-Time Uniform Ciphertext Security

An encryption scheme is one-time uniform ciphertext secure if $\forall m \in \mathscr{M}$,

$$D_0 = \left\{ \mathsf{ct} : \begin{matrix} k \leftarrow \mathsf{KeyGen}() \\ \mathsf{ct} \leftarrow \mathsf{Enc}(k, m) \end{matrix} \right\} \quad \equiv \quad D_1 = \left\{ \mathsf{ct} : \mathsf{ct} \xleftarrow{\$} \mathscr{C} \right\}$$

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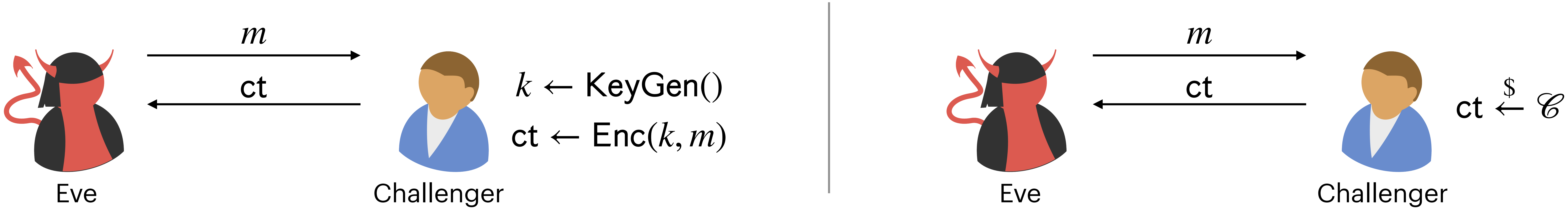
Eve's view when Alice encrypts m

What we want Eve's view to look like

Security: D_1 carries no information about the message

Alternative View of One-Time Uniform Ciphertext Security

Consider the following two interactions between Eve and a challenger.



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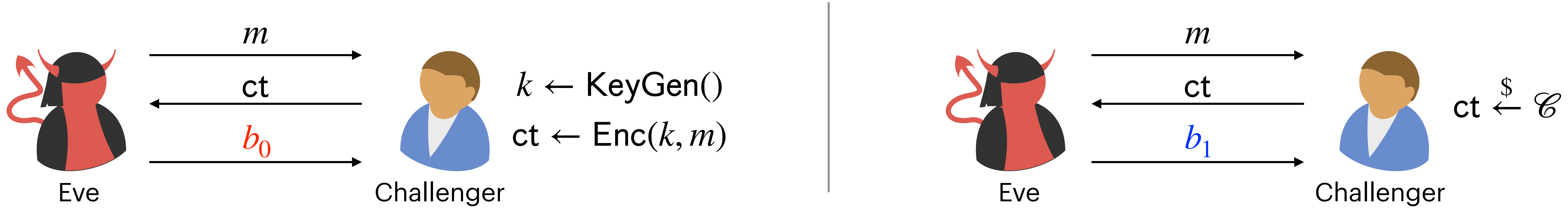

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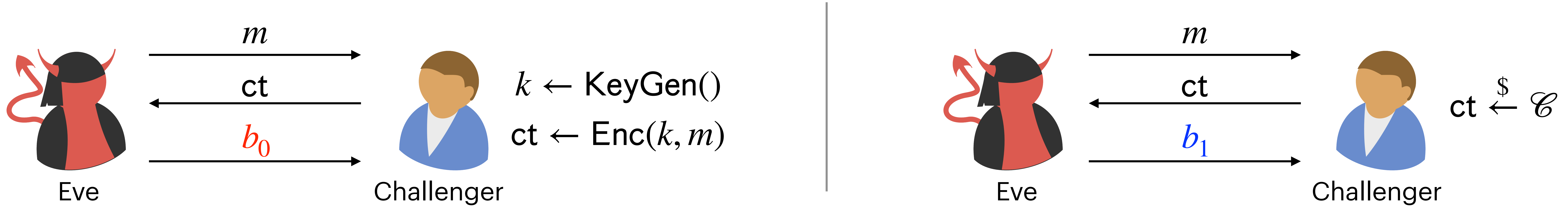

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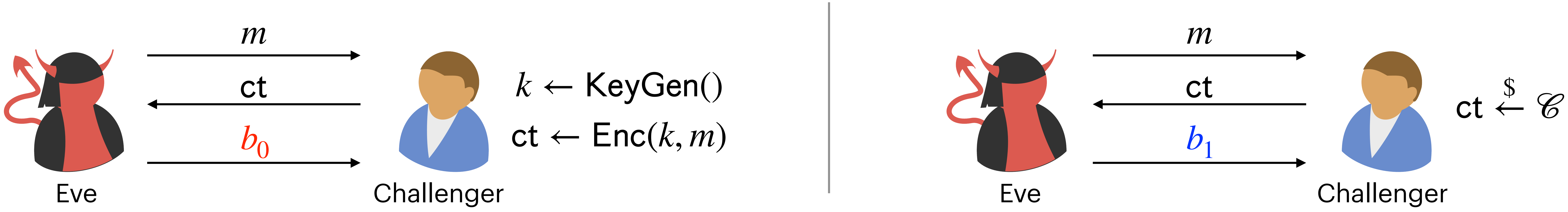

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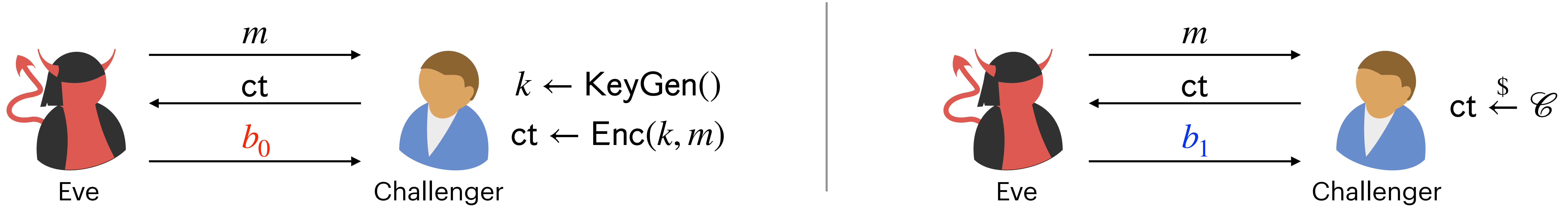

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- Eve is allowed to choose the plaintext. If the scheme is secure when Eve chooses the plaintext, it is secure when she has only partial information about the plaintext.

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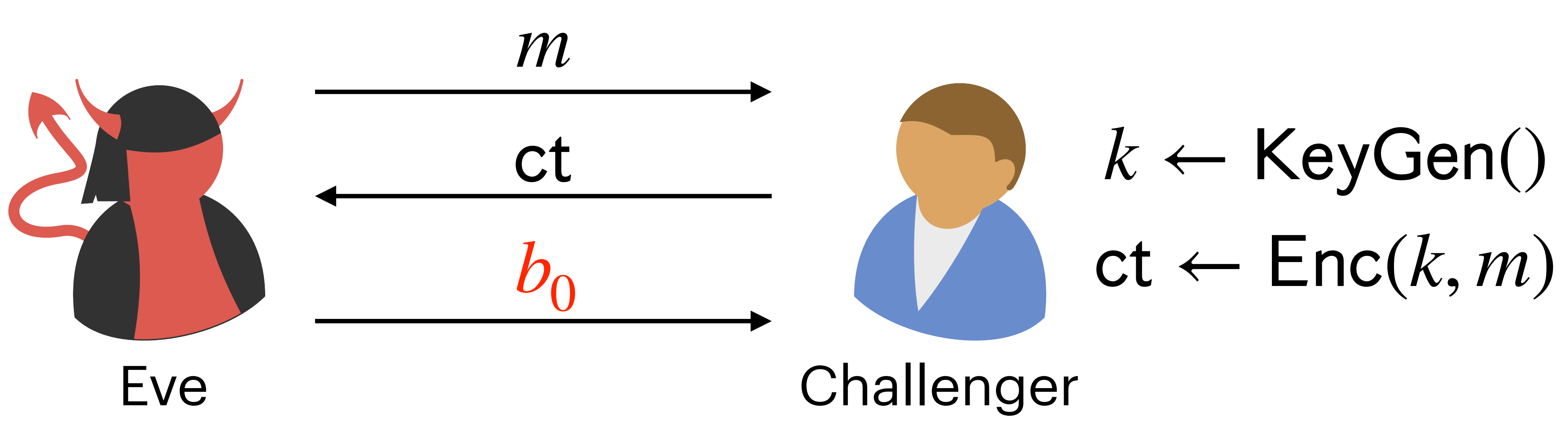

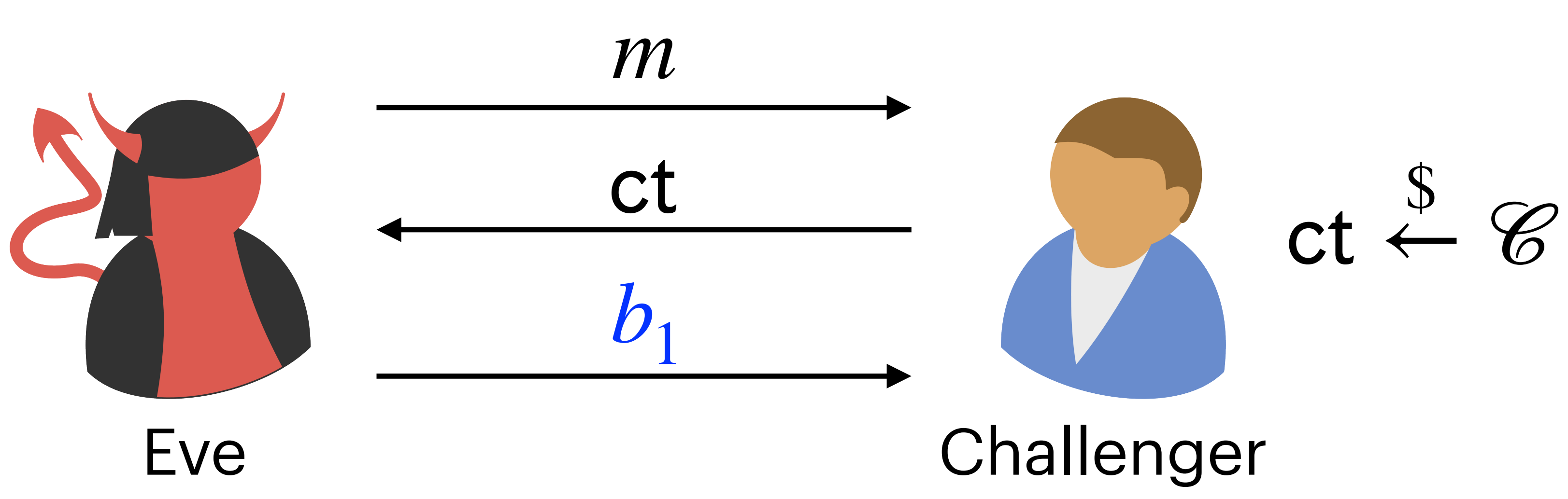

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- Interaction with a challenger helps model what Eve can see during encryption and what remains hidden.
- Eve is allowed to choose the plaintext. If the scheme is secure when Eve chooses the plaintext, it is secure when she has only partial information about the plaintext.
- Equivalent to the previous definition of one-time uniform ciphertext security.

Recap: One-Time Pad

One-Time Pad

Let λ be a positive integer and let $\mathscr{K} = \mathscr{M} = \mathscr{C} = \{0,1\}^\lambda$.

- KeyGen(): $k \xleftarrow{\$} \{0,1\}^\lambda$.
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Theorem: One-time pad is correct and has one-time uniform ciphertext security.

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- An alternative idea for defining security of encryption schemes.
  - The secret key should be kept hidden from Eve.
  - The key is only used to encrypt one plaintext.
  - ~~The ciphertext looks uniformly random to Eve.~~ Encryptions of m_0 look like encryptions of m_1 to Eve.

(One-Time) Perfect Security

An encryption scheme is one-time perfectly secure if $\forall m_0, m_1 \in \mathcal{M}$,

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Security: The ciphertext distribution is independent of the message.

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Corollary: One-time pad is perfectly secure.

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Perfect security exactly captures our intuition.

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Extends immediately to t-message perfect security: $|\mathcal{K}| \geq |\mathcal{M}|^t$.

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- Since there are $|\mathscr{M}|$ messages, there must be at least $|\mathscr{M}|$ keys. Thus, $|\mathscr{K}| \geq |\mathscr{M}|$.

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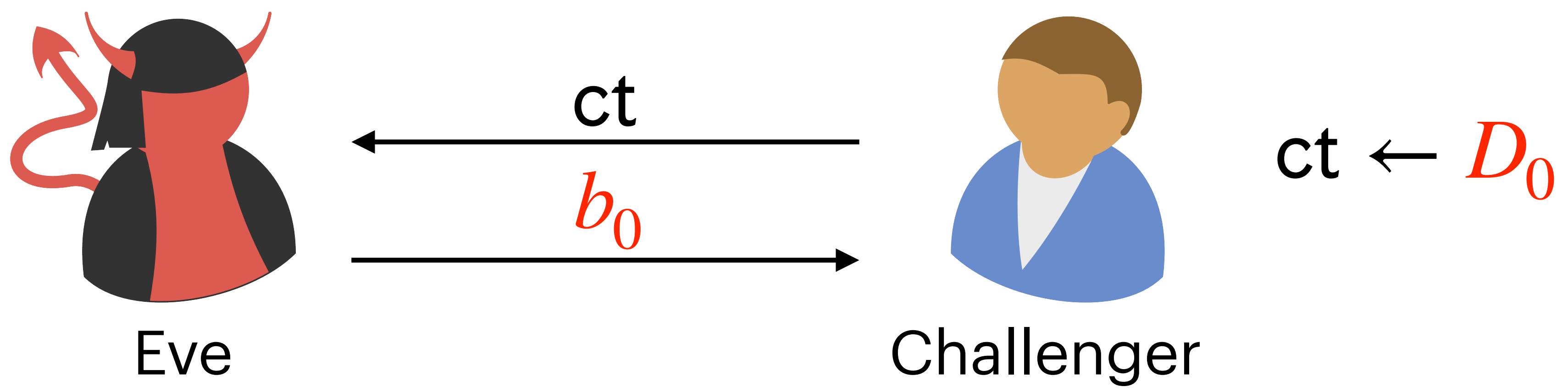

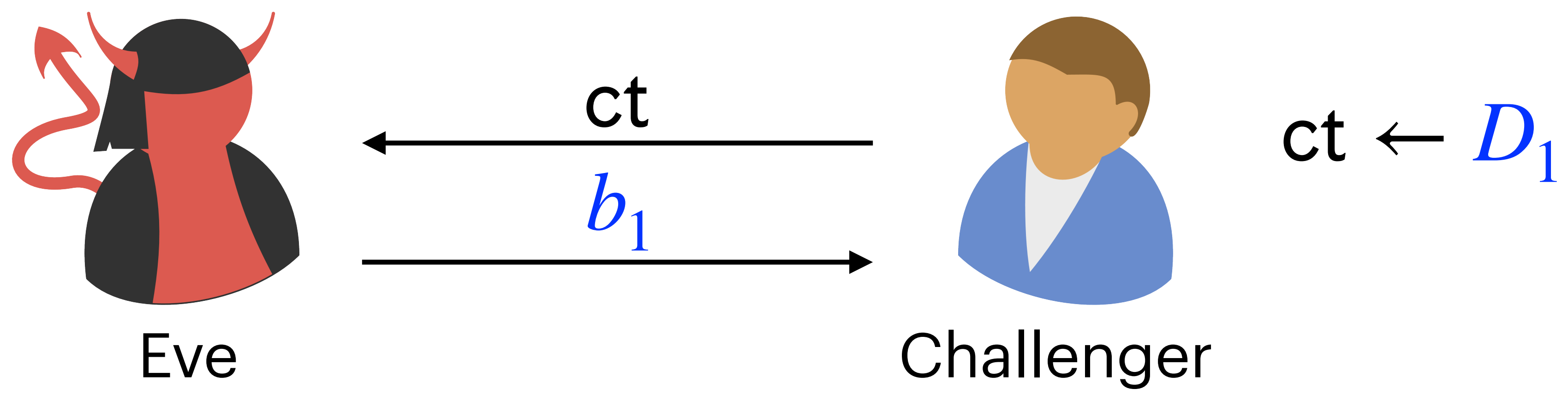

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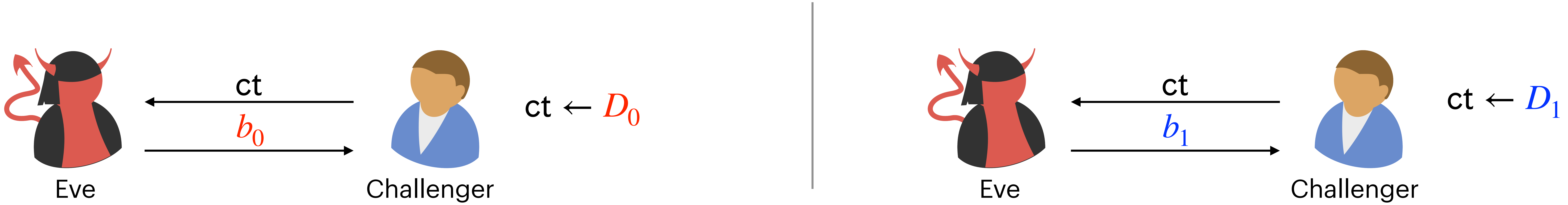

$D_0 \overset{\epsilon}{\approx} D_1$ if

$$|\Pr[b_0 = 1] - \Pr[b_1 = 1]| \leq \epsilon$$

where the probability is over the randomness of KeyGen and Enc.

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Shannon's theorem can be extended to show that statistically secure encryption schemes still require long keys.

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1/8	011	001
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1/8	101	111
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- What if we relax security to only hold against attacks that are feasible to carry out?

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