Computational Security

601.442/642 Modern Cryptography

29th January 2026

Announcement

- Homework 1 due **today**.
- Homework 2 will be out today and will be due next Thursday (5th Feb).

Recap: Limitations of Perfect Security

Theorem (Shannon): Any perfectly secure encryption scheme with key space $\mathscr{K}$ and message space $\mathscr{M}$ satisfies

$$|\mathscr{K}| \geq |\mathscr{M}|.$$

Perfect security is too strong. Can we weaken the definition?

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2^{-28}	Winning this week's Powerball jackpot
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  - Are brute force attacks feasible?

Cost of Computation

- One way to measure the cost of computation is through the monetary value required to carry it out.

CPU Cycles	Approx. Cost	Reference
2^{50}	\$3.50	Cup of coffee
2^{55}	\$100	Tickets to Portland Trailblazers game
2^{65}	\$130,000	Median home price in Oshkosh, WI
2^{75}	\$130 million	Average budget of one of the Harry Potter movies
2^{92}	\$20 trillion	GDP of the United States
2^{99}	\$2 quadrillion	All human economic activity since 300,000 BC
2^{128}	???	A billion human civilizations' worth of effort

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      Don't worry about the adversary blindly guessing the key!
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A scheme is (T, ϵ)-secure if any adversary running for time at most T succeeds in breaking the scheme with probability at most ϵ.

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(T, ϵ)-Computational Indistinguishability

Two distributions X and Y are (T, ϵ)-computationally indistinguishable if for all adversaries A that run in time at most T,

$$\left| \Pr_{x \leftarrow X}\left[A(x) = 1\right] - \Pr_{y \leftarrow Y}\left[A(y) = 1\right] \right| \leq \epsilon,$$

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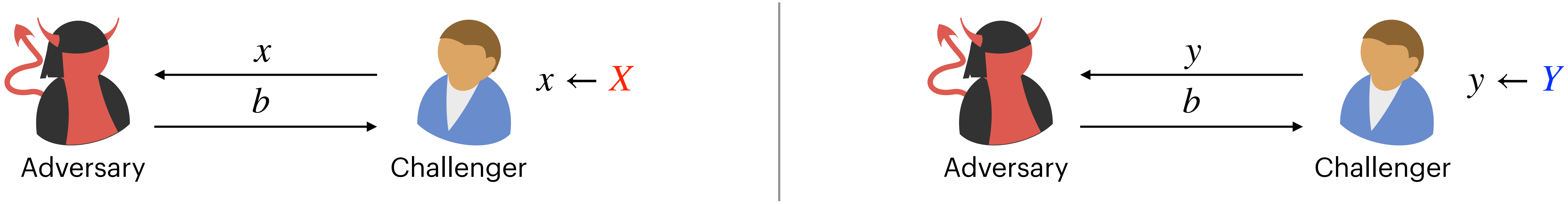

Adversary cannot tell X and Y apart except with small probability.

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(T, ϵ)-Computational Security

An encryption scheme is (T, ϵ)-computationally secure if for all $m_0, m_1 \in \mathcal{M}$, the following distributions are (T, ϵ)-computationally indistinguishable:

$$D_0 = \left\{ \mathsf{ct} : \begin{array}{c} k \leftarrow \mathsf{KeyGen}() \\ \mathsf{ct} \leftarrow \mathsf{Enc}(k, m_0) \end{array} \right\} \qquad D_1 = \left\{ \mathsf{ct} : \begin{array}{c} k \leftarrow \mathsf{KeyGen}() \\ \mathsf{ct} \leftarrow \mathsf{Enc}(k, m_1) \end{array} \right\}$$

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    - **Analogy:** We consider asymptotic growth in runtime for sorting algorithms; not their runtime on lists of 10,000 values i.e., we have a “knob” to tune the runtime for lists of different length.

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  - **8GHz Computers with $\lambda = 160$:** Encryption takes 3.2 seconds. Adversary that runs for ~13 weeks can break security with probability at most 2^{-80}!

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  - Primitives have a fixed (small) polynomial runtime and the adversary can run for much longer (arbitrary polynomial runtime).

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For poly-time algorithms, events that occur with negligible probability look like they never occur.

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- Let $\Lambda = \max(\Lambda_f, \Lambda_g, 2)$. For all $\lambda > \Lambda$ we have

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Ensembles

Our goal is to give an asymptotic definition of computational indistinguishability.

<u>(T, ϵ)-**Computational Indistinguishability**</u>

Two distributions X and Y are (T, ϵ)-computationally indistinguishable if for all adversaries A that run in time at most T,

$$\left| \Pr_{x \leftarrow X}\left[A(x) = 1\right] - \Pr_{y \leftarrow Y}\left[A(y) = 1\right] \right| \leq \epsilon,$$

where the probability is over sampling from the distributions X and Y, and the randomness of A.

- It is not very meaningful to talk about individual distributions when we want to capture asymptotic behavior.
- For example, using longer keys leads to distributions over longer bit strings.

Ensembles

Probability Ensemble

Let $\mathscr{I}$ be a countable index set. An ensemble indexed by $\mathscr{I}$ is a sequence of random variables $\{X_i\}_{i \in \mathscr{I}}$.

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- An ensemble is simply sequence of random variables $X_1, X_2, \ldots$
  - Allows us to focus on asymptotic behavior of distributions e.g., what happens when the key is a sufficiently long, uniformly random bit string.

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Two probability ensembles $X = \{X_i\}_{i \in \mathbb{N}}$ and $Y = \{Y_i\}_{i \in \mathbb{N}}$ are computationally indistinguishable if

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Denotes string of λ ones.

Ensures A is polynomial in λ.

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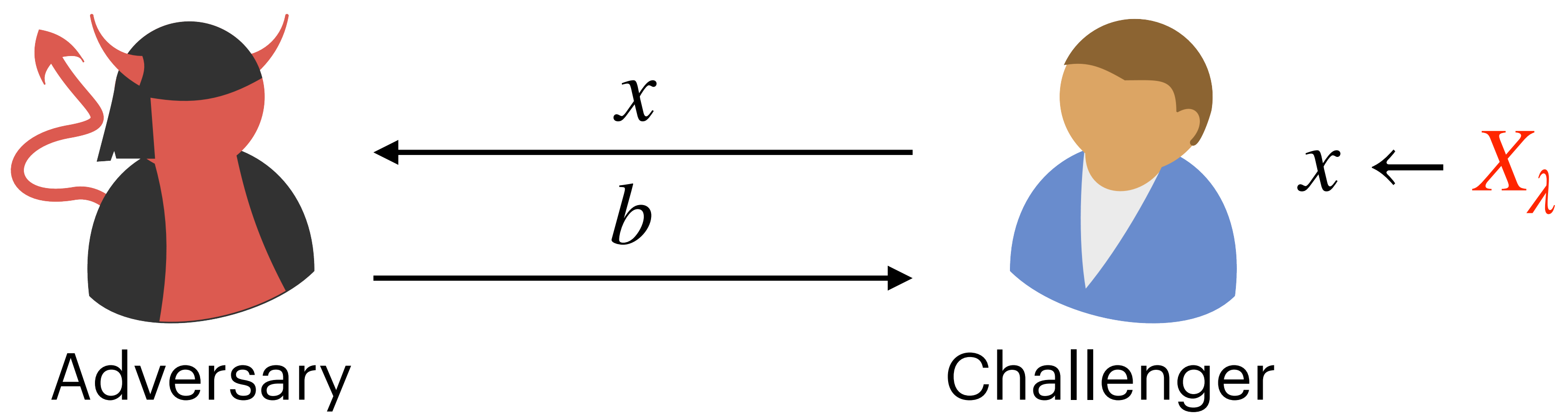

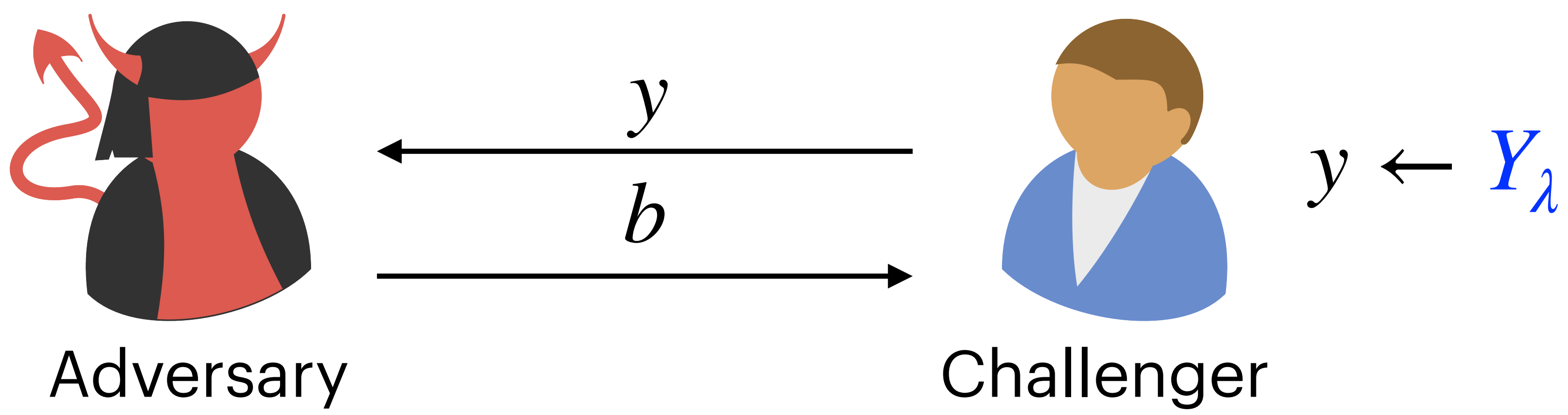

No efficient test can distinguish between the ensembles X and Y.

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- We use $X \overset{c}{\approx} Y$ as a shorthand to denote that the two ensembles are computationally indistinguishable.

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is called the adversary's **advantage** in distinguishing between X and Y.

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- $X \overset{c}{\approx} Y$ if all non-uniform PPT adversaries have negligible advantage in distinguishing between the two ensembles.