Pseudorandomness II

601.442/642 Modern Cryptography

5th February 2026

Announcement

- Homework 2 is due **today**.
- Homework 3 will be out today and due next Thursday (12th Feb).

Recap: Pseudorandom Generator

<u>**Pseudorandom Generator**</u>

A deterministic algorithm G is called a pseudorandom generator (PRG) if:

- G can be computed in polynomial time,
- On input any $s \in \{0,1\}^\lambda$, G outputs a $\ell(\lambda)$-bit string such that $\ell(\lambda) > \lambda$,
- $\{G(s) : s \xleftarrow{\$} \{0,1\}^\lambda\} \quad \overset{c}{\approx} \quad \{r : r \xleftarrow{\$} \{0,1\}^{\ell(\lambda)}\}$

The stretch of G is defined as $\ell(\lambda) - \lambda$.

Recap: One-Time Computational Security

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$$D_0 = \left\{ \mathsf{ct} : \begin{array}{c} k \leftarrow \mathsf{KeyGen}(1^\lambda) \\ \mathsf{ct} \leftarrow \mathsf{Enc}(k, m_0) \end{array} \right\} \overset{c}{\approx} D_1 = \left\{ \mathsf{ct} : \begin{array}{c} k \leftarrow \mathsf{KeyGen}(1^\lambda) \\ \mathsf{ct} \leftarrow \mathsf{Enc}(k, m_1) \end{array} \right\}$$

Recap: Pseudorandom OTP

Pseudorandom One-Time Pad

Let λ be the security parameter and $\ell(\lambda)$ be a polynomial. Let G be a PRG with stretch $\ell(\lambda) - \lambda$.

- KeyGen(1^λ): $k \xleftarrow{\$} \{0,1\}^\lambda$.
- Enc(k, m): ct $:= G(k) \oplus m$.
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Keys are shorter than the message: λ-bit keys and $\ell(\lambda)$-bit messages.

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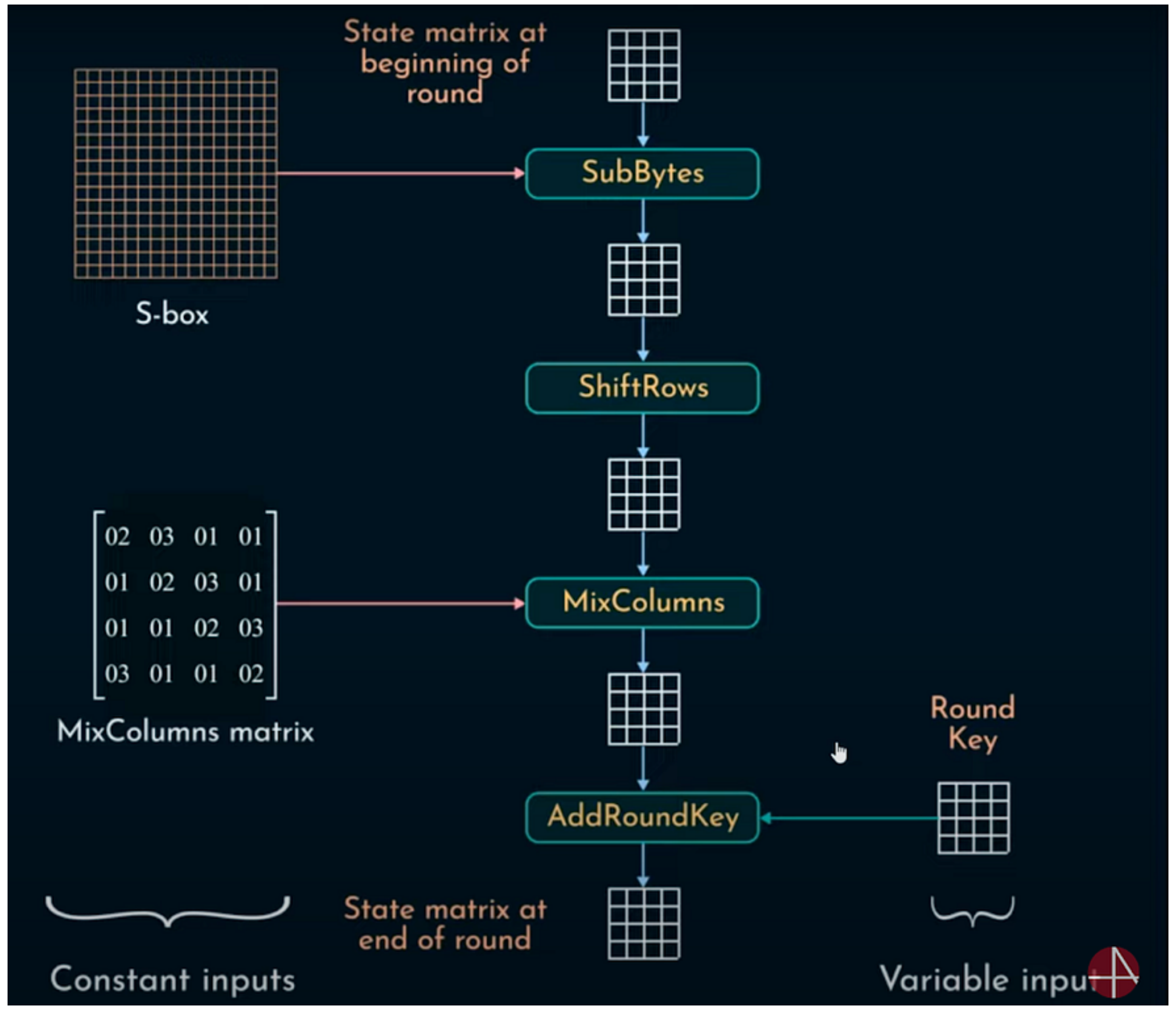

nesoacademy.org

Single round of AES

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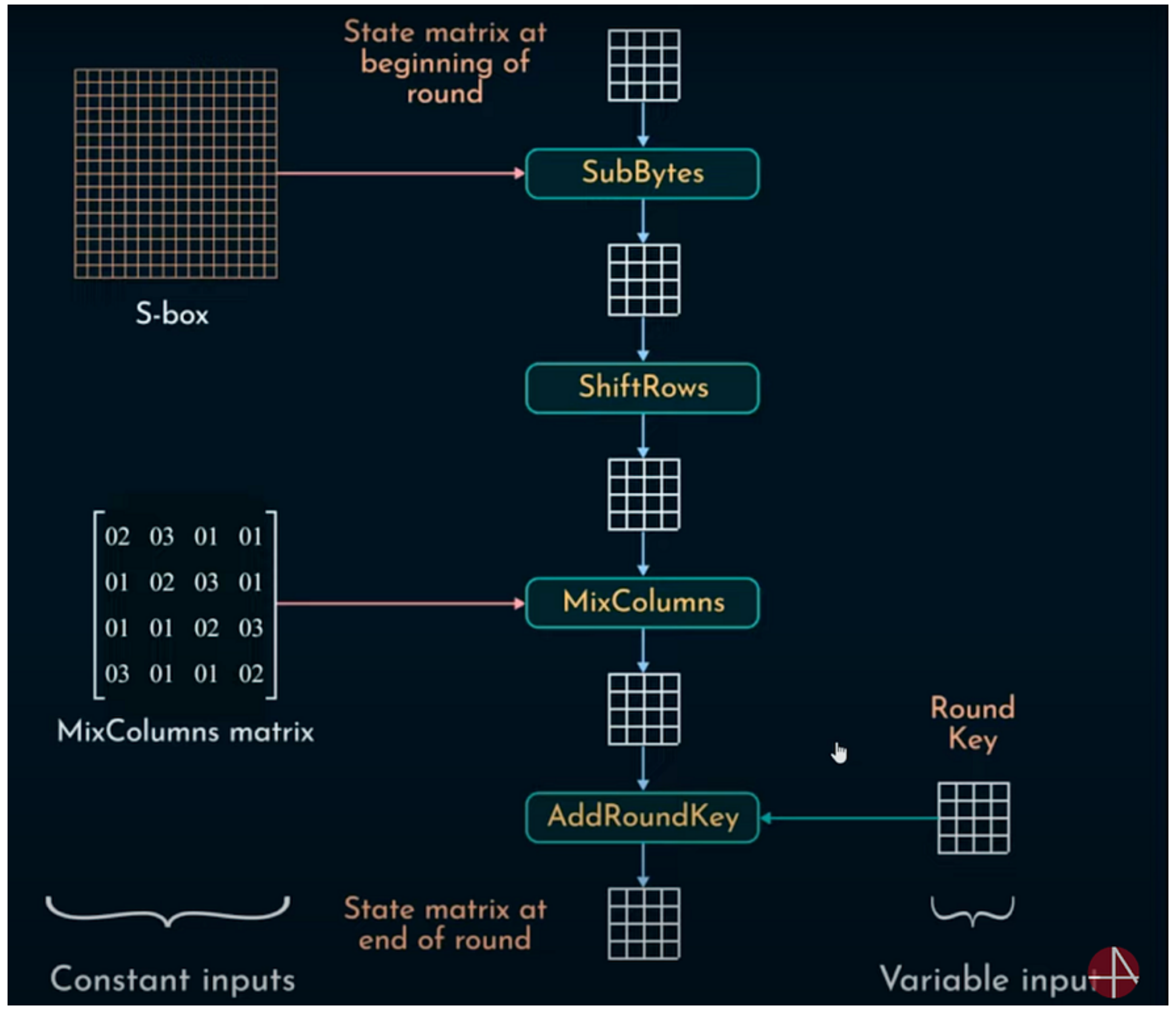
    - Do extensive cryptanalysis.



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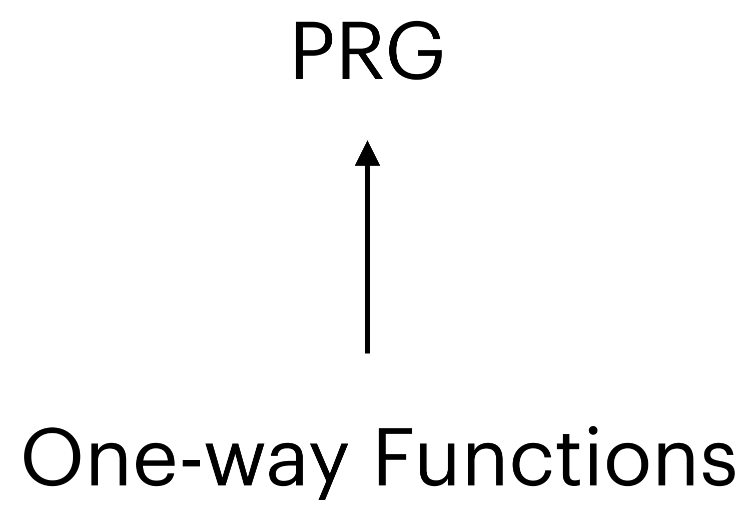

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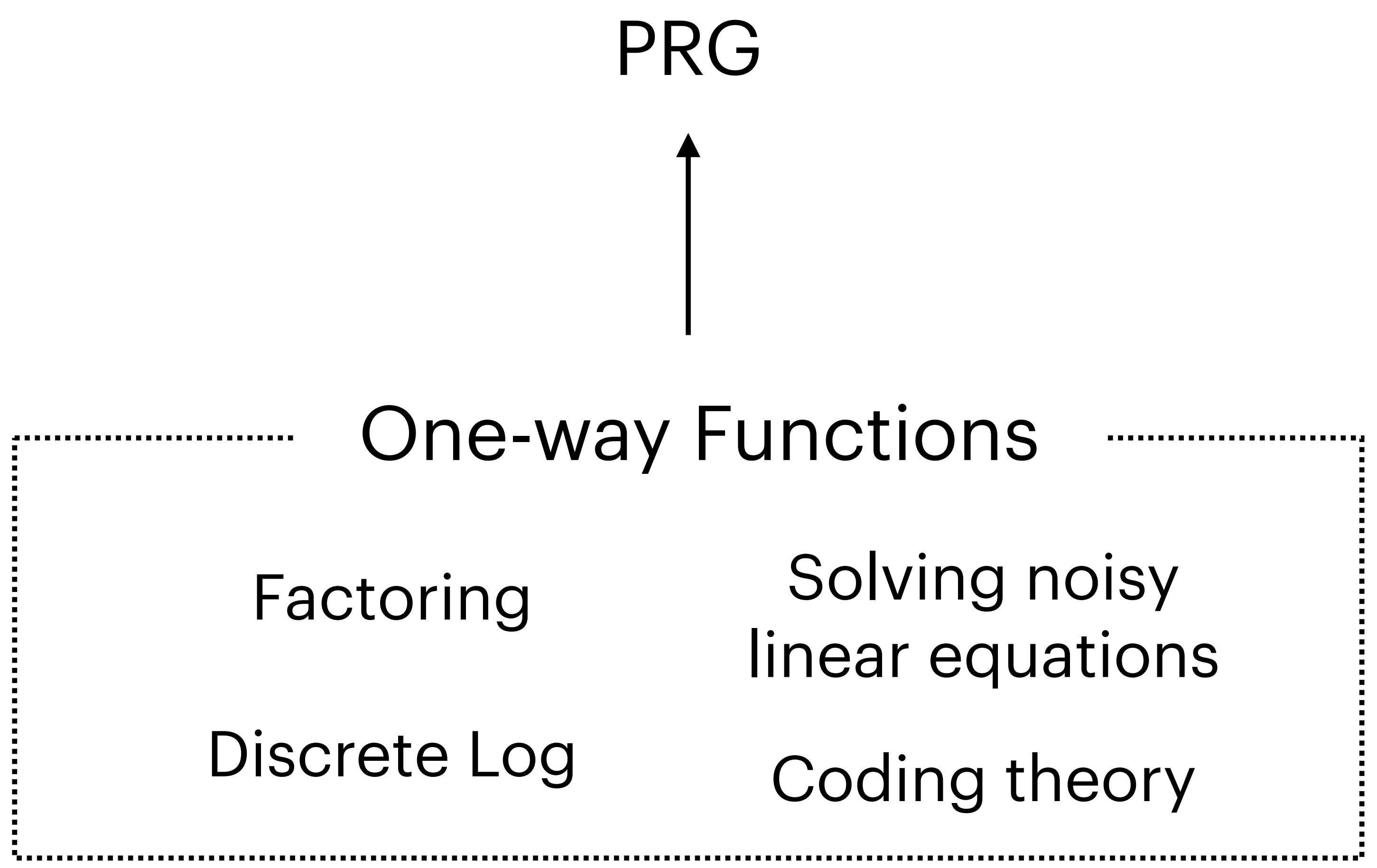

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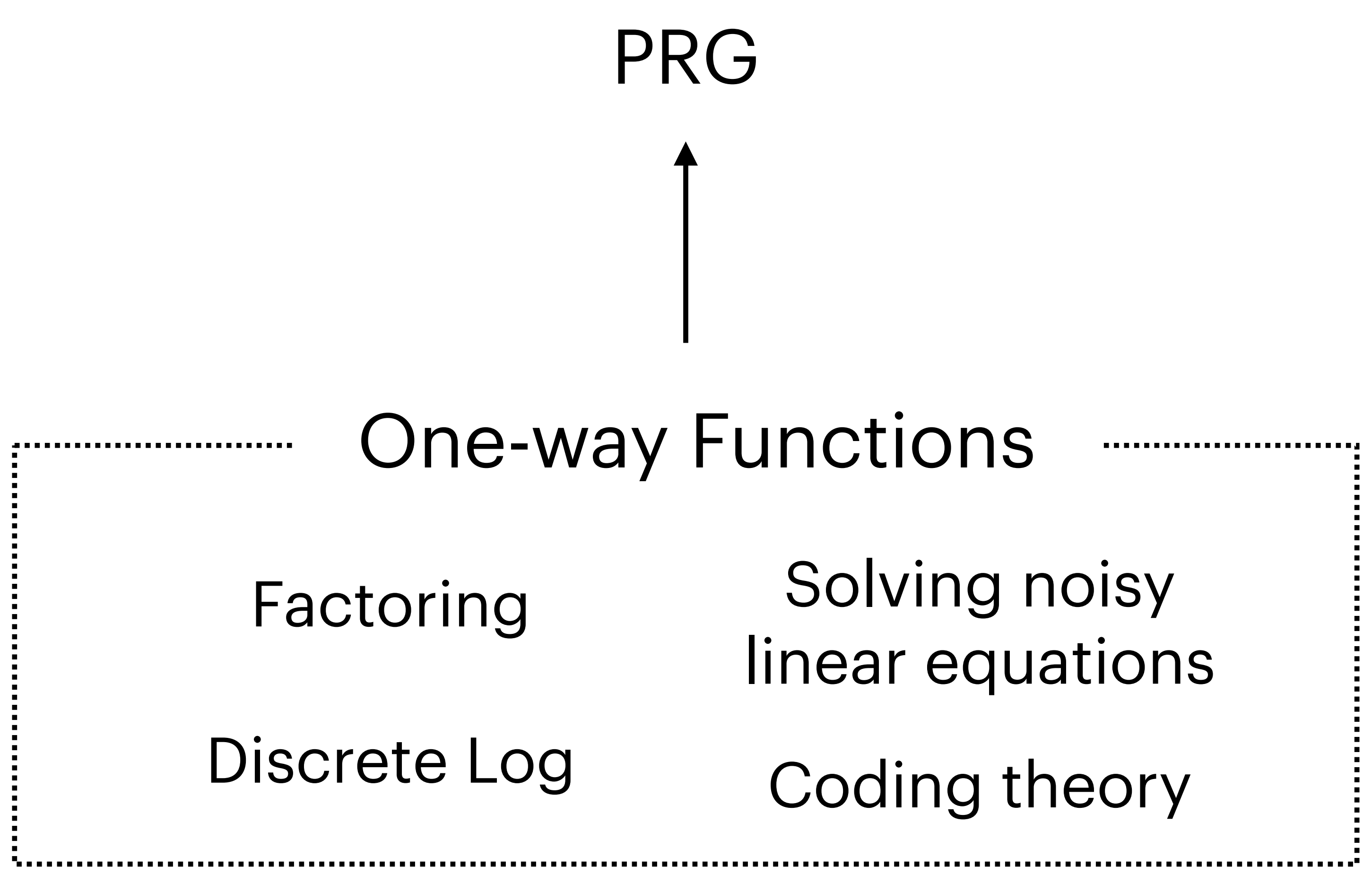
  - We will focus on the foundational methodology in this course.



Pseudorandom OTP

Pseudorandom One-Time Pad

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A PRG with 1-bit stretch implies a PRG with arbitrary polynomial-bit stretch.

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Suffices to show that $\forall i \in \{0,\ldots,\ell-1\}$, $H_i \overset{c}{\approx} H_{i+1}$. Why?

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Reduction!

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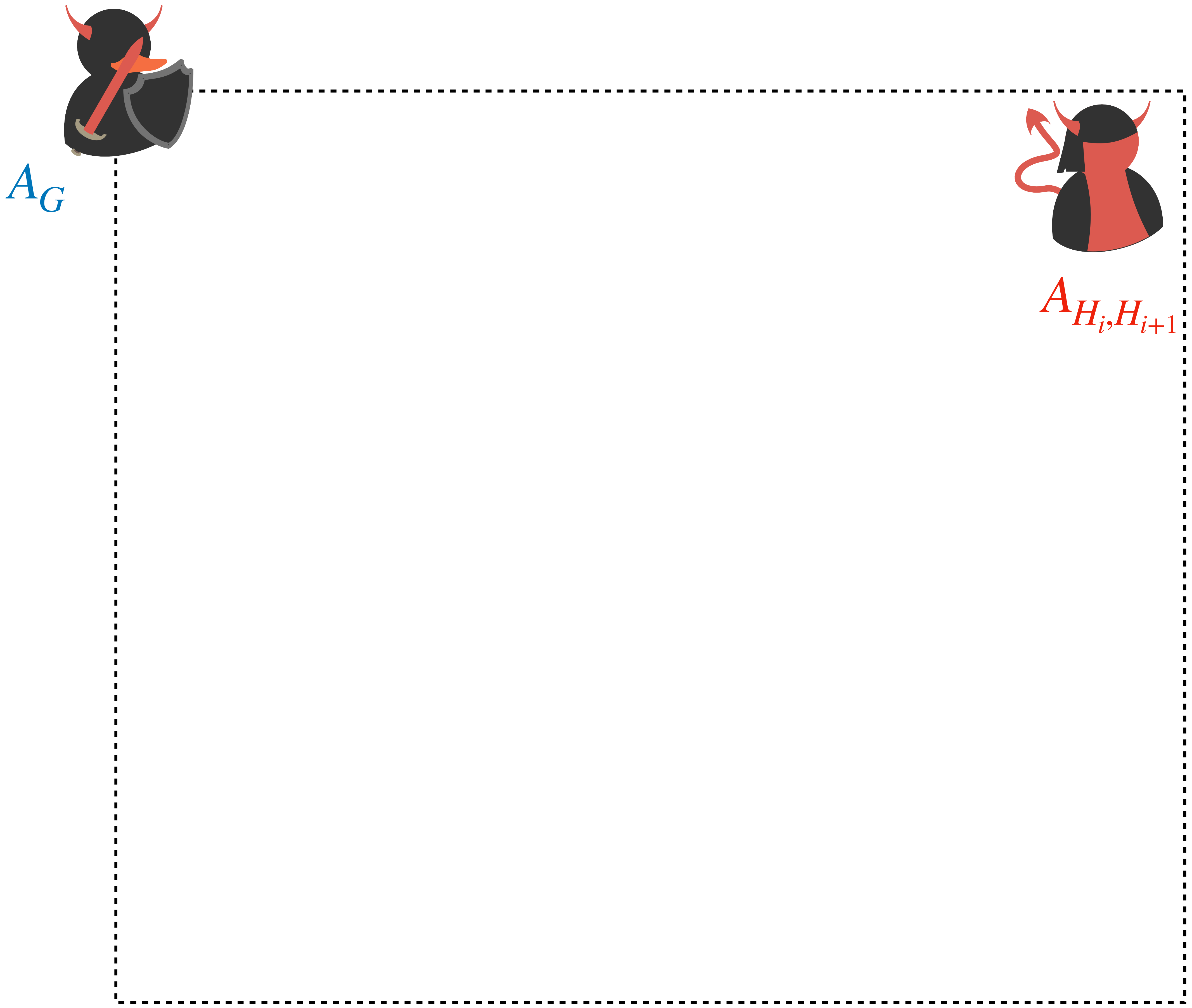

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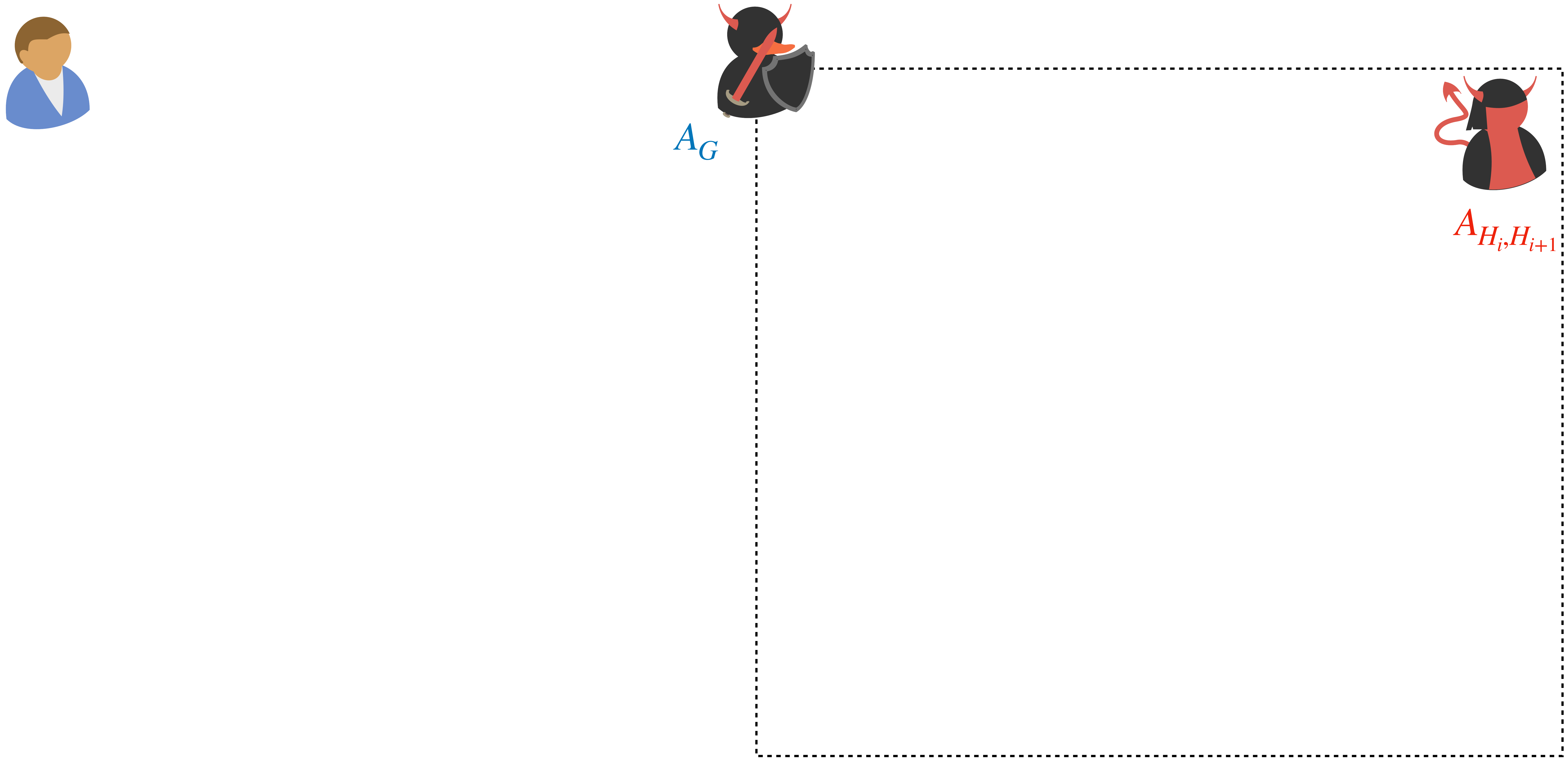

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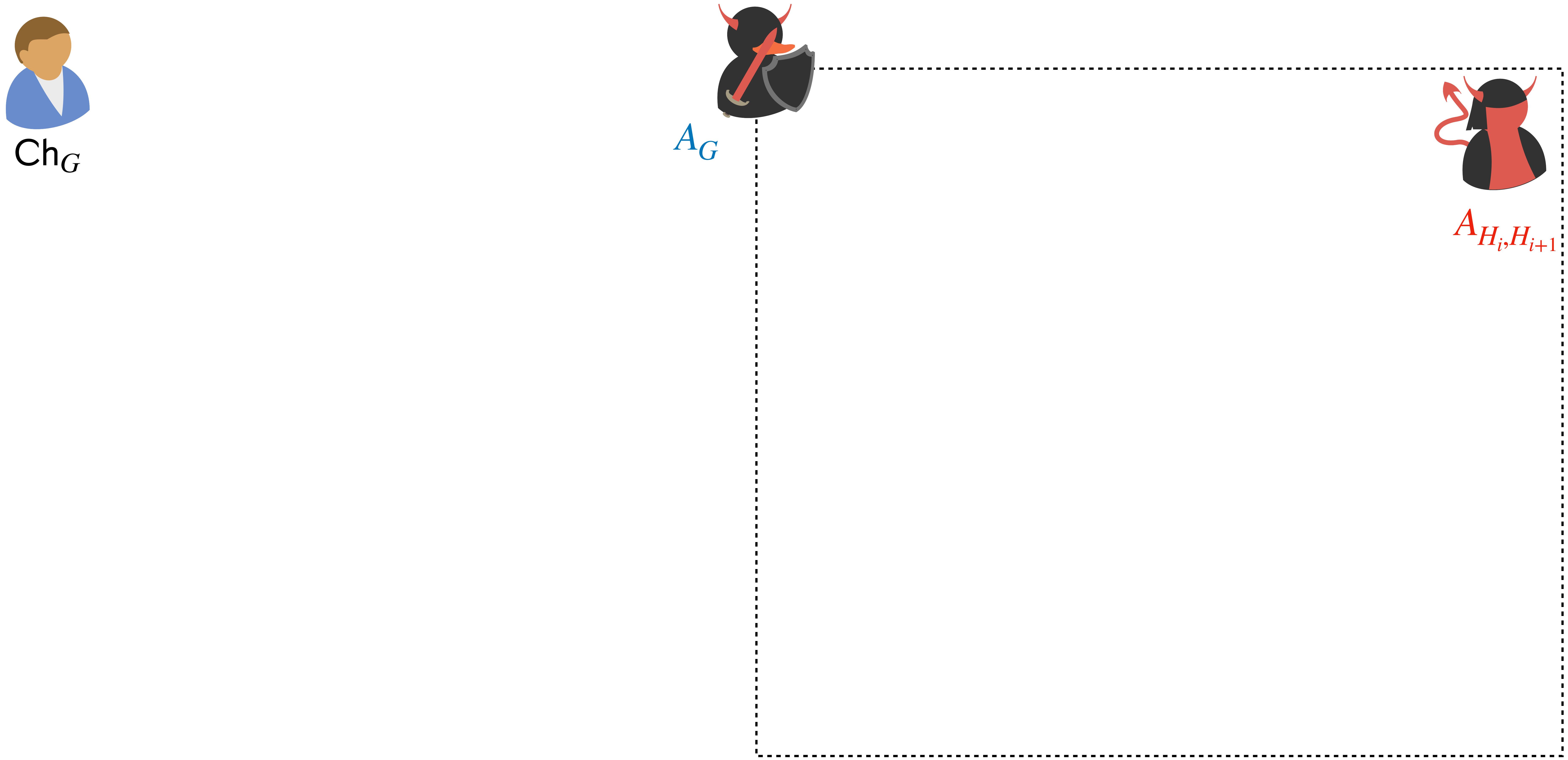

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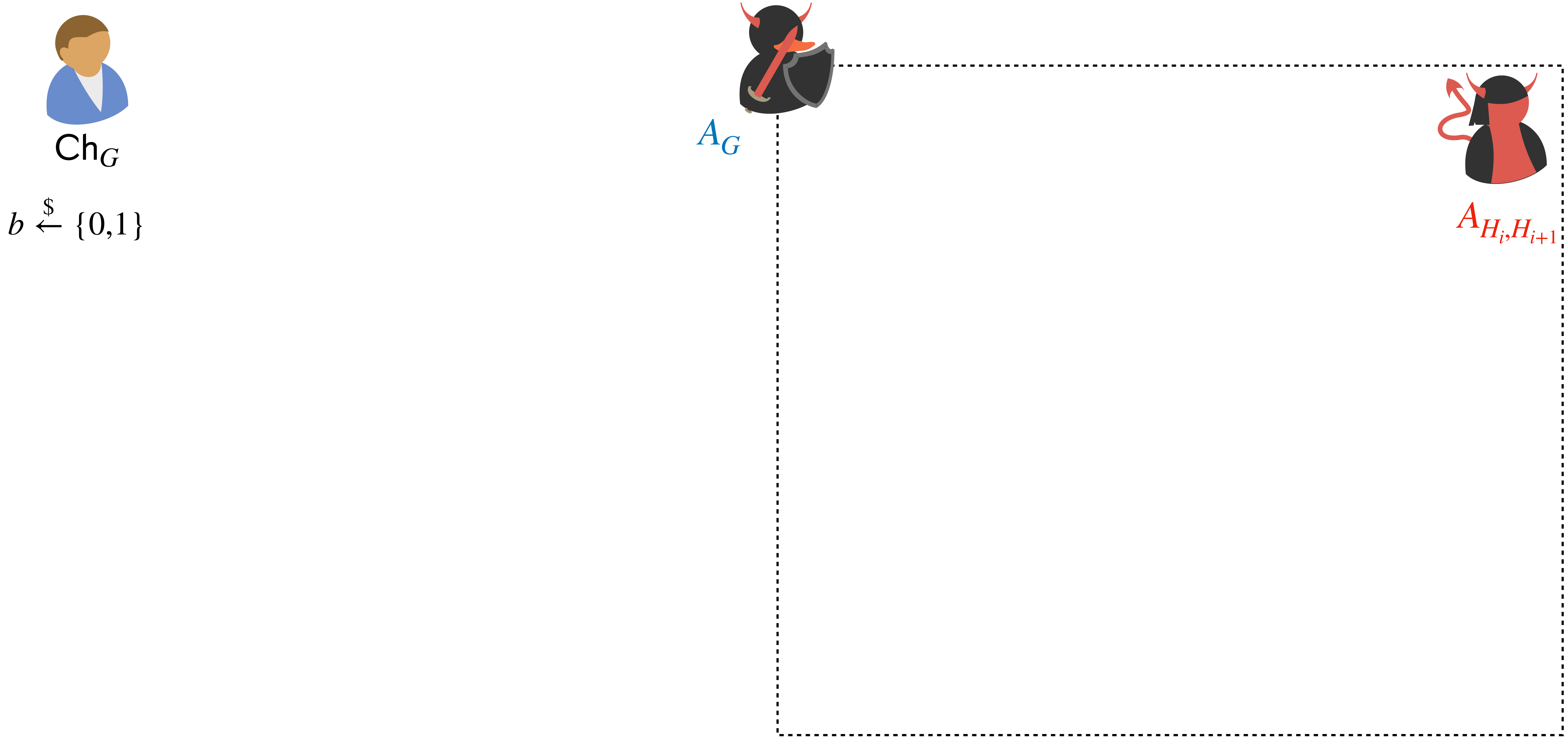

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$b \xleftarrow{\$} \{0,1\}$

If $b = 0$:

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A_G

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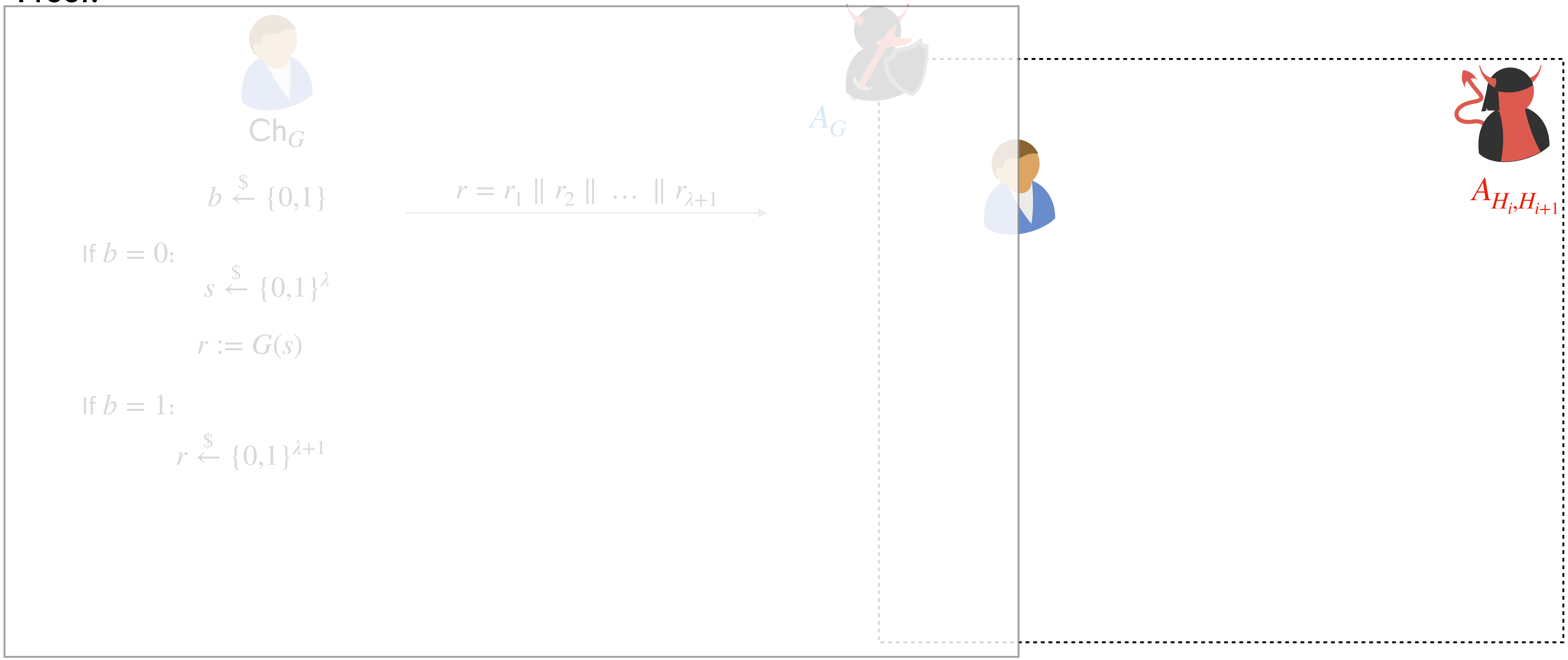

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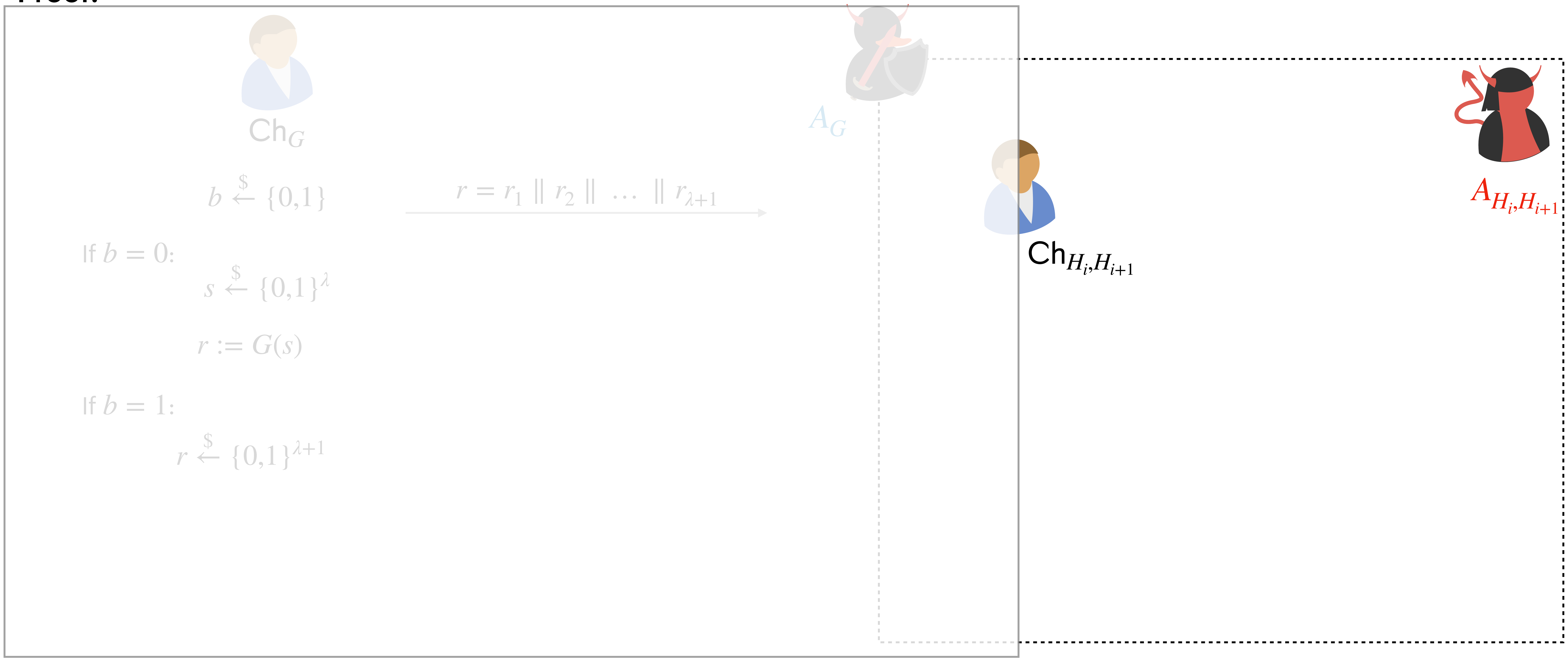

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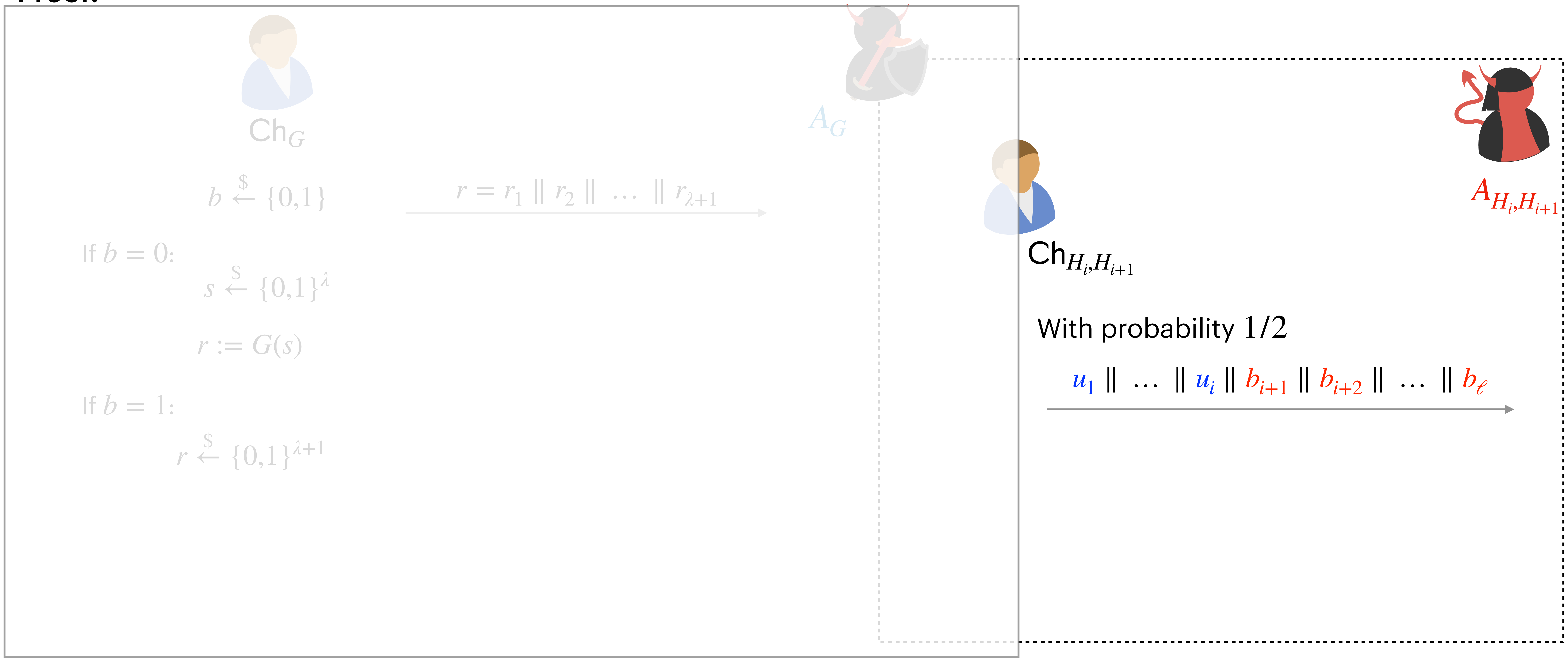

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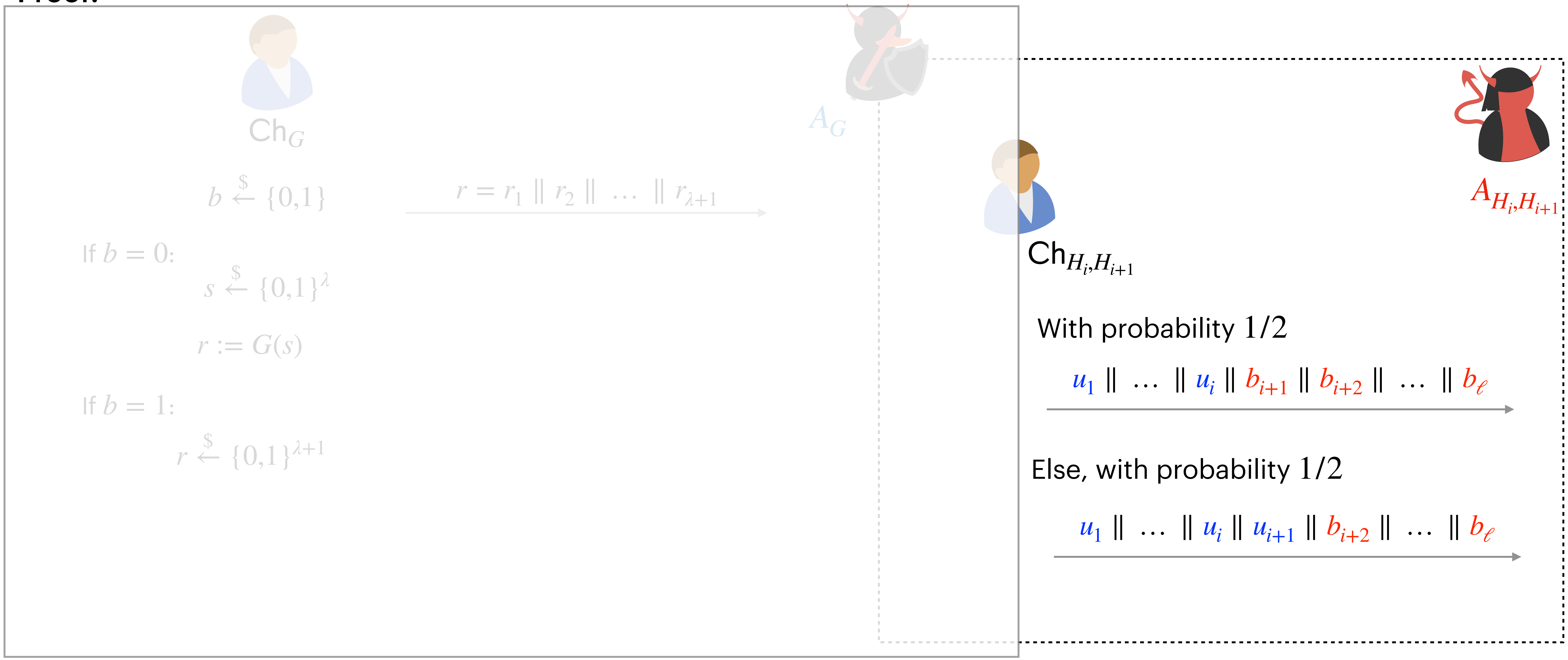

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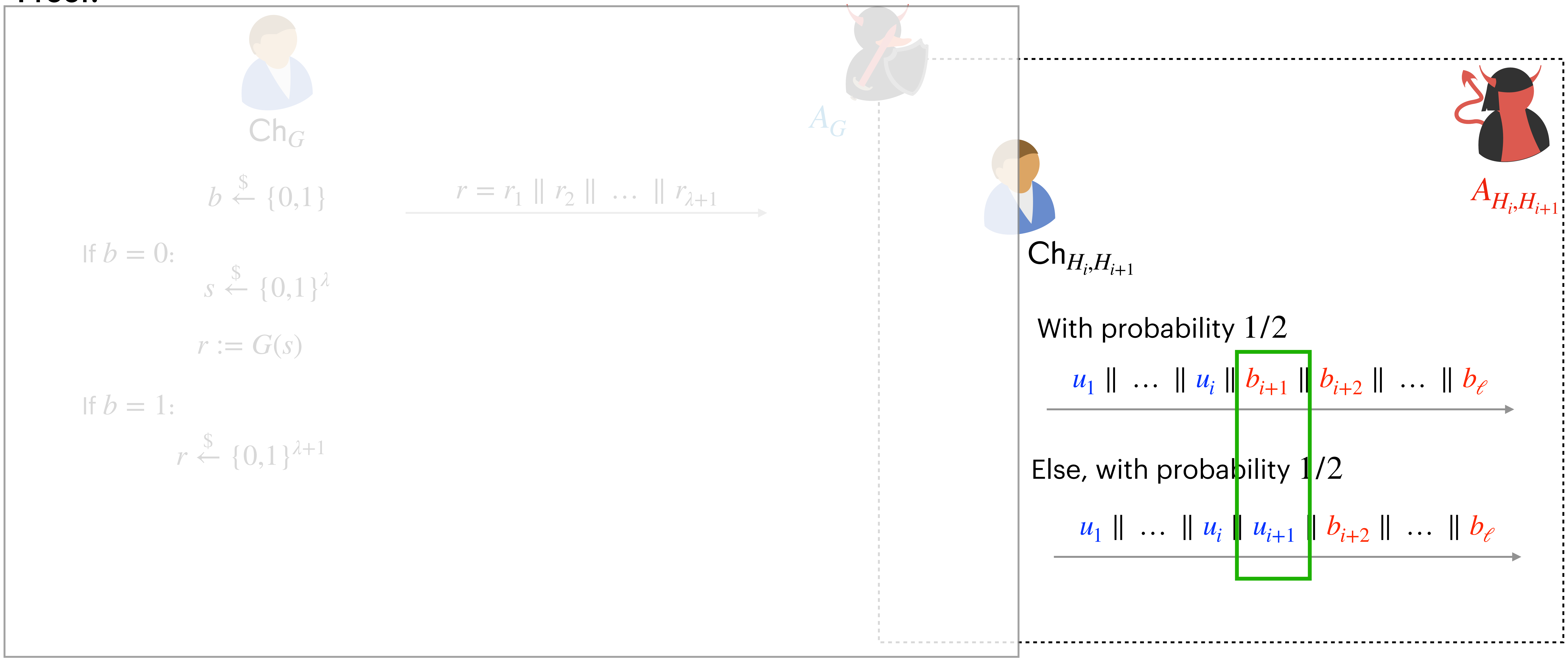

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$u_1 \| u_2 \| \ldots \| u_i \| r_{\lambda+1} \| b_{i+2} \| \ldots \| b_\ell$ →

Input mapping:

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Input mapping:

If $b = 0$, $r_{\lambda+1}$ is output of G and $A_{H_i,H_{i+1}}$'s view is identical to H_i.

PRG Length Extension: Proof

Claim: If G is a PRG then for each $i \in \{0,\ldots,\ell-1\}$, $H_i \overset{c}{\approx} H_{i+1}$.

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$b \overset{\$}{\leftarrow} \{0,1\}$

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$u_1 \parallel u_2 \parallel \ldots \parallel u_i \parallel r_{\lambda+1} \parallel b_{i+2} \parallel \ldots \parallel b_\ell$ →

Input mapping:

If $b = 0$, $r_{\lambda+1}$ is output of G and $A_{H_i,H_{i+1}}$'s view is identical to H_i.

If $b = 1$, $r_{\lambda+1}$ is uniformly random and $A_{H_i,H_{i+1}}$'s view is identical to H_{i+1}.

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← b'

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Output mapping:

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← b'

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Output mapping:

A_G has the same advantage as $A_{H_i,H_{i+1}}$.

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Claim: If G is a PRG then G_{poly} is a PRG.

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Therefore, from the hybrid lemma

If for all efficient adversaries A $\left| \Pr_{k \overset{\$}{\leftarrow} \{0,1\}^\lambda} \left[A(1^\lambda, G(k)) = 1 \right] - \Pr_{r \overset{\$}{\leftarrow} \{0,1\}^{\lambda+1}} \left[A(1^\lambda, r) = 1 \right] \right| \leq \mathsf{negl}(\lambda)$

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- **Concrete Security:** If G is (T, ϵ)-secure, then G_{poly} is $(T, \ell(\lambda) \cdot \epsilon)$ secure.

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Claim: If G is a PRG then G_{poly} is a PRG.

Proof:

Claim: If G is a PRG then for each $i \in \{0,\ldots,\ell-1\}$, $H_i \overset{c}{\approx} H_{i+1}$.

Therefore, from the hybrid lemma

If for all efficient adversaries A $\left|\Pr_{k \xleftarrow{\$} \{0,1\}^\lambda}\left[A(1^\lambda, G(k)) = 1\right] - \Pr_{r \xleftarrow{\$} \{0,1\}^{\lambda+1}}\left[A(1^\lambda, r) = 1\right]\right| \leq \text{negl}(\lambda)$

Then, for all efficient adversaries A $\left|\Pr_{k \xleftarrow{\$} \{0,1\}^\lambda}\left[A(1^\lambda, G_{\text{poly}}(k)) = 1\right] - \Pr_{r \xleftarrow{\$} \{0,1\}^{\ell(\lambda)}}\left[A(1^\lambda, r) = 1\right]\right| \leq \ell(\lambda) \cdot \text{negl}(\lambda)$

- **Concrete Security:** If G is (T, ϵ)-secure, then G_{poly} is $(T, \ell(\lambda) \cdot \epsilon)$ secure.
  - **Example:** If $\epsilon = 2^{-40}$ then encrypting a 131 KB message using G_{poly} leads to $\epsilon_{\text{poly}} = 2^{-20}$.

PRG Length Extension

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Probability (ϵ)	Event
2^{-10}	Full house in 5-card poker
2^{-20}	Royal flush in 5-card poker
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2^{-40}	Royal flush in two consecutive poker games
2^{-60}	Next meteorite that hits Earth lands on this slide

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  - This is fine for an asymptotic approach.
  - However, the loss matters in practice for concrete security.

Multi-Message Security

One-Time Computational Security

An encryption scheme with message length $\ell := \ell(\lambda)$ is one-time computationally secure if $\forall m_0, m_1 \in \{0,1\}^\ell$

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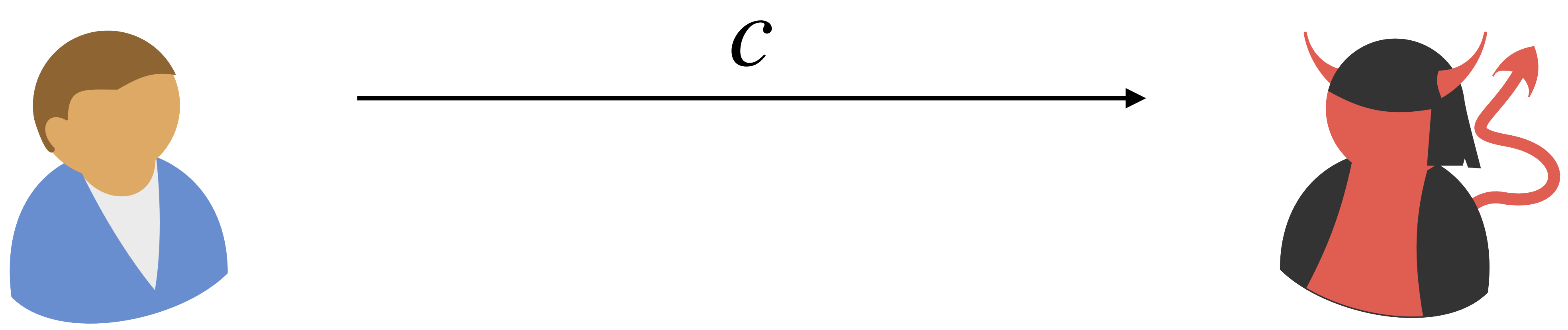

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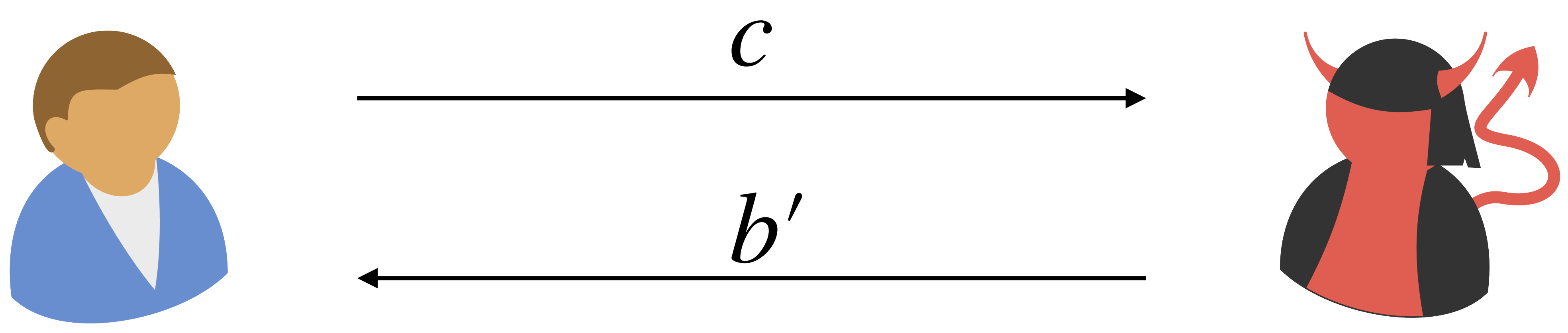

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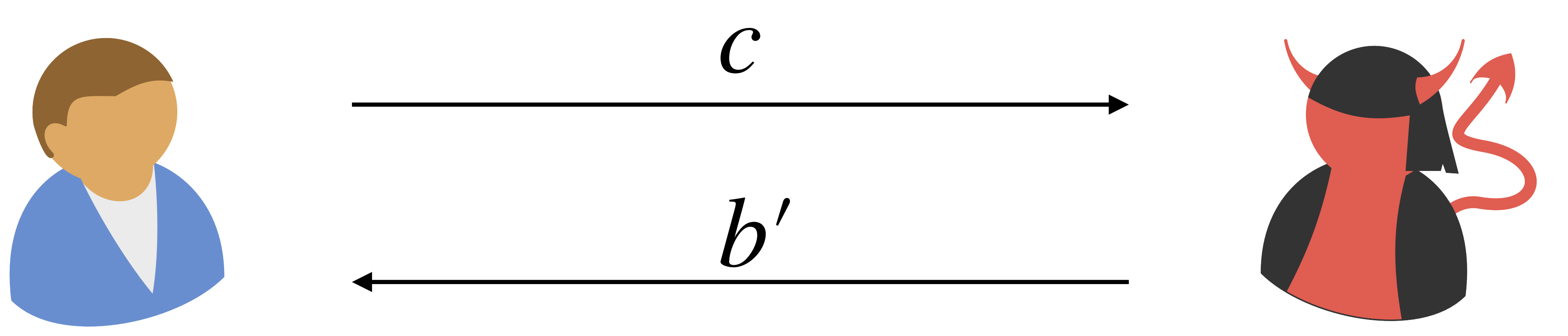

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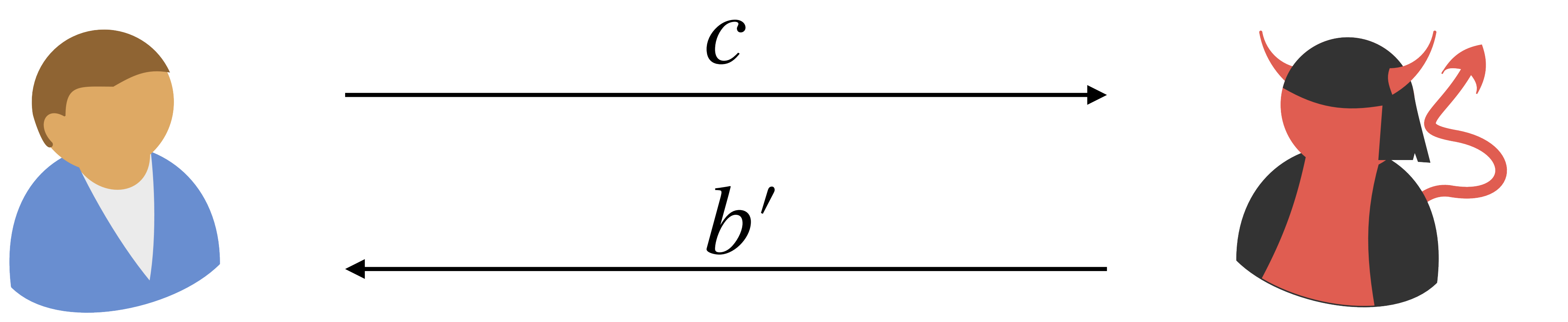

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$\forall \mathscr{A}, \forall m_0, m_1,$

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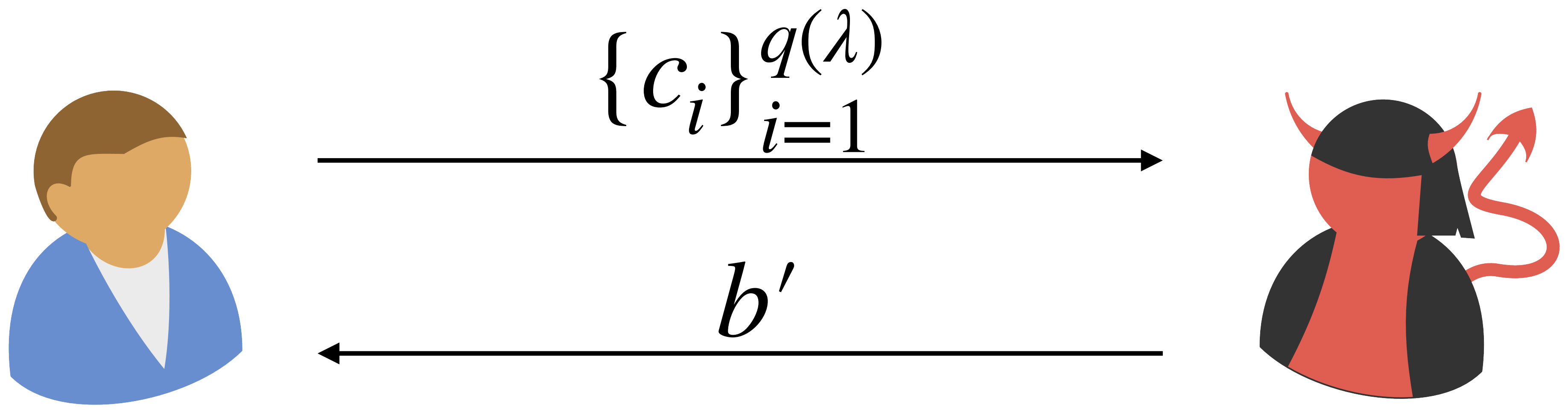

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Does Pseudorandom OTP satisfy this definition?

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Idea: Can we design a multi-message secure encryption scheme that is **stateful**?

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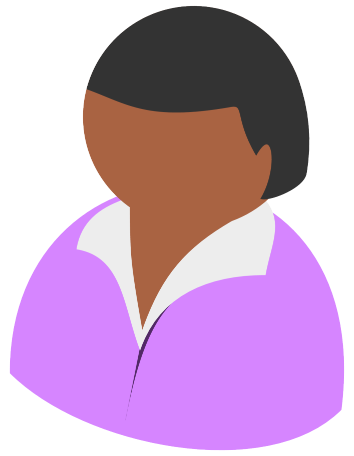

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k

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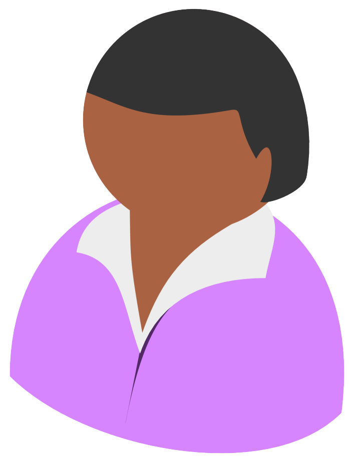


Stateful Multi-Message Secure Encryption

We just saw a PRG that can output a *polynomial* number of pseudorandom bits.

What if we use one chunk at a time?

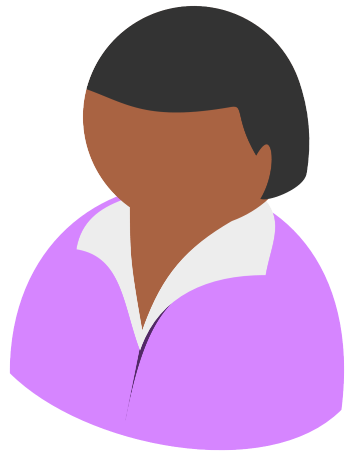
$G(s)[0]$ $G(s)[1]$ $G(s)[2]$

$$G(s) = 00010101 \ldots 11100010 \ldots 01011010 \ldots \rightarrow \text{poly}(\lambda)$$

k

$i = 0$

k



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$\longrightarrow$

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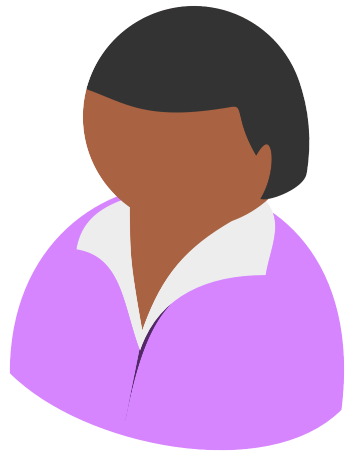

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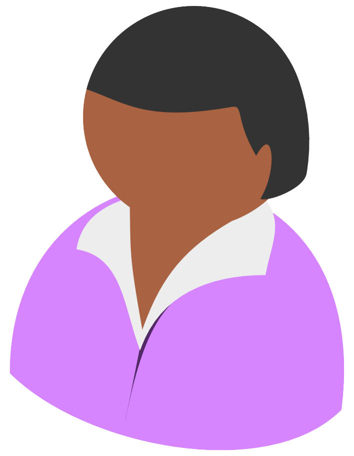

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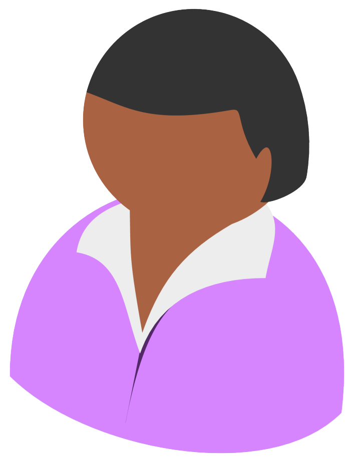

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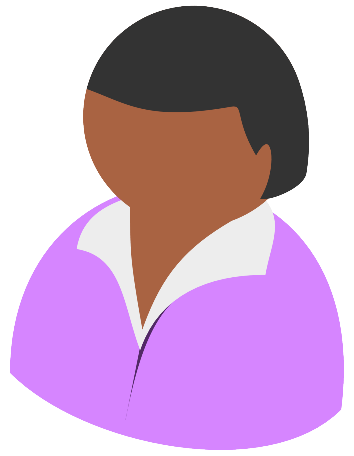

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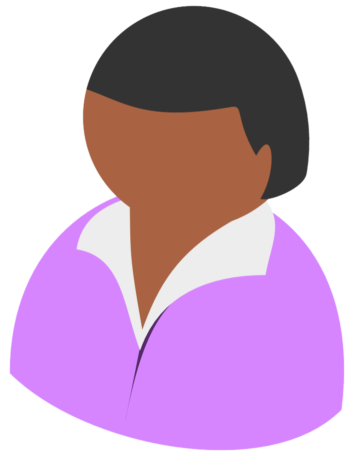

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What are the downsides of keeping state?

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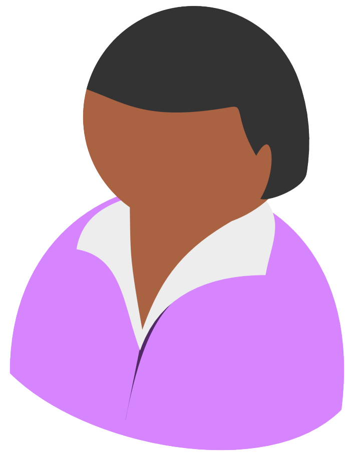

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Losing it!

k

$i = 1$

$c_0 = m_0 \oplus G(k)[0]$ →

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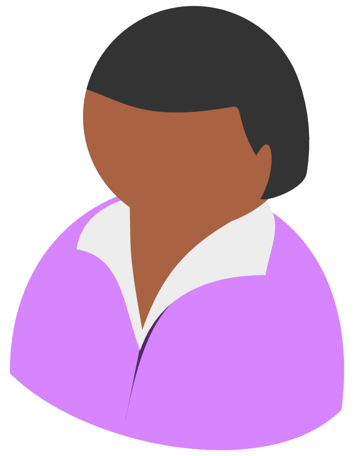

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Stateless Multi-Message Secure Encryption

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What if we remove state by randomly sampling the chunk index?

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$i_0 \xleftarrow{\$} [0\ldots\text{poly}(\lambda)]$

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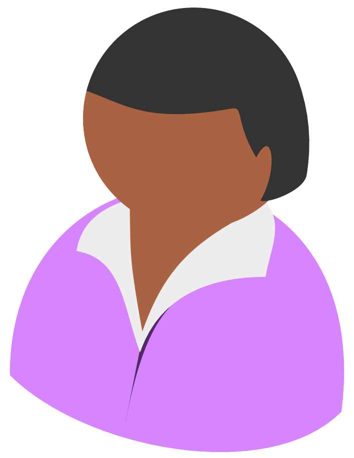

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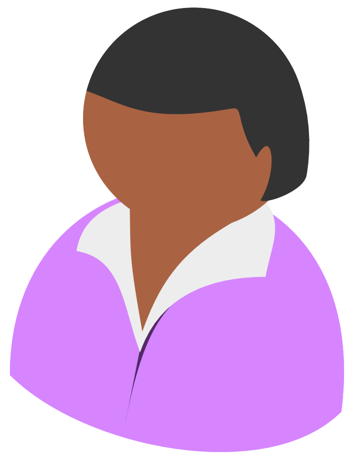

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What happens if $i_0 = i_1$?

What is the probability that $i_0 = i_1$?

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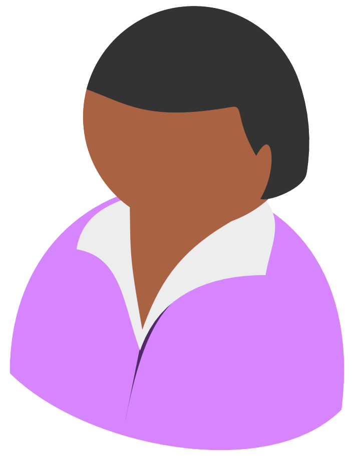

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This is totally insecure! There is a non-negligible chance of sampling the same index, and so a non-negligible chance of reusing a chunk!

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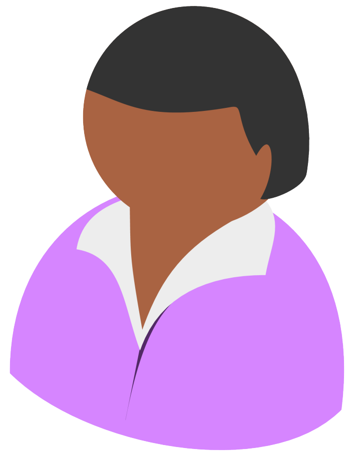

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Idea: What if we could index into an *exponential* amount of randomness?

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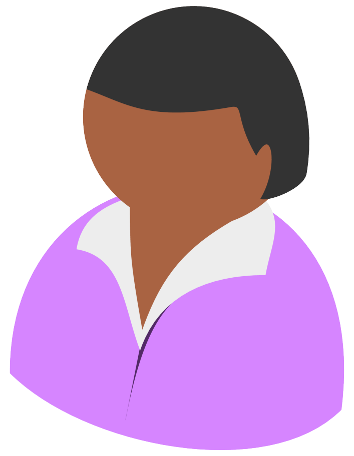

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What if Alice and Bob shared an *exponential* amount of randomness?

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$F =$

x	r
000…000	11001010
000…001	10011111
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$\cdots$

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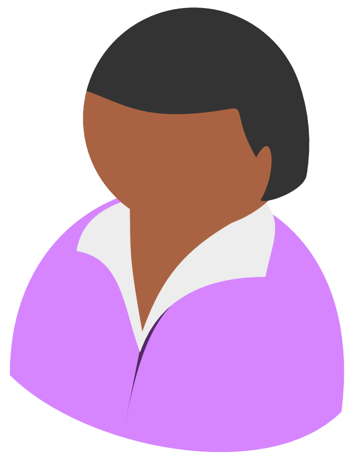

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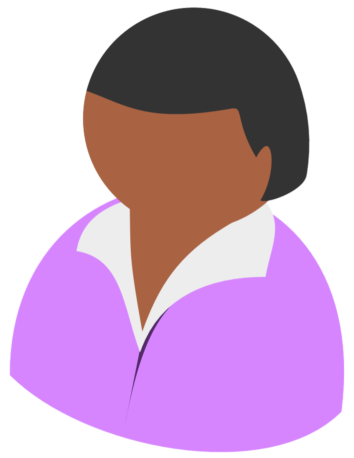

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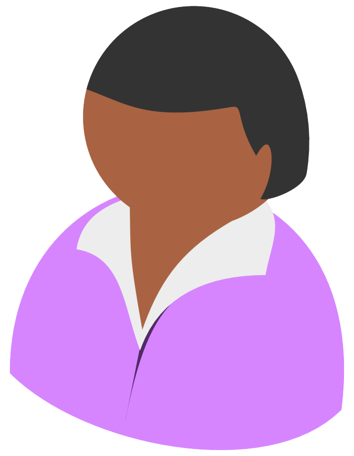

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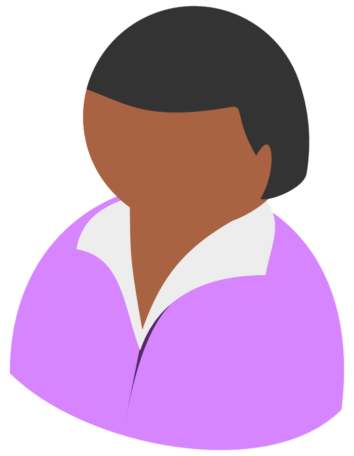

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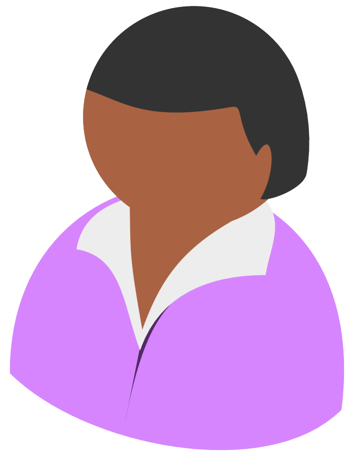

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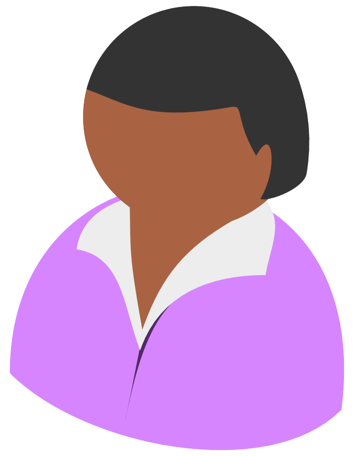

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F is a *random function*

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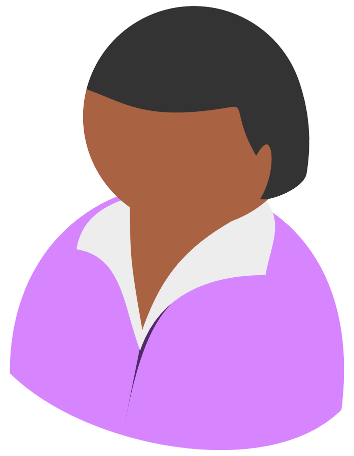

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